

A Risk Assessment Method Based on a Novel Distance-Include Aggregation Operator in the Probabilistic Uncertain Linguistic T-Spherical Fuzzy Sets Environment

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ARTICLE INFO	ABSTRACT
Article history: Received 2 July 2026 Received in revised form 10 August 2026 Accepted 21 September 2026 Available online 3 October 2026 Keywords: Distance measures; Multi-attribute decision making; FMEA; PULTSF; Decision ranking	Traditional Failure Mode and Effects Analysis (FMEA) can suffer from information loss, inappropriate weighting, and limited discrimination among risk priorities. To address these limitations, this paper proposes the Probabilistic Uncertain Linguistic T-Spherical Fuzzy Set (PULTSFS) and develops a multi-attribute decision-making model based on the PULTSFDIOWA operator. The operator uses the distance between each failure mode and the positive ideal solution as an inducing variable and incorporates different weighting strategies and distance measures to aggregate risk assessment information. This approach accounts for both decision preferences and the distribution of assessment data while representing the fuzziness, hesitancy, and neutrality of expert evaluations. The model is evaluated through an illustrative example, comparative analysis, and sensitivity analysis. The results indicate improved discrimination among failure modes and greater ranking robustness compared with the traditional methods considered in the study. The proposed framework provides an approach to prioritising failure modes in complex engineering systems under uncertain expert assessments.

1. Introduction

Failure Mode and Effects Analysis (FMEA) is a systematic risk assessment tool whose primary objective is to identify potential failure modes and their effects on systems or products [1]. It has been widely applied in engineering manufacturing, healthcare, transportation, and other fields with high requirements for safety and reliability [2, 3]. Traditional FMEA relies on the Risk Priority Number (RPN) to evaluate the risk levels of failure modes, calculating and ranking risk values through the product of three risk factors—Occurrence (O), Severity (S), and Detection (D)—based on which strategies are formulated to reduce the probability of these potential failures [4, 5]. However, this traditional calculation method suffers from defects that cannot be overlooked [6, 7]:

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- i. Loss of evaluation information. Through the product operation $RPN = S \times O \times D$, traditional FMEA directly loses the original detailed information at the level of each risk attribute in expert evaluations, resulting in an overly coarse characterization of risk.
- ii. Erroneous assignment of identical weights. Traditional RPN implicitly assigns equal weights to the three risk factors, whereas S, O, and D are not equally important in practice, and thus their true differences in importance cannot be reflected.
- iii. Concealment of differences in risk characteristics. RPN assigns identical weights to risk factors (RFs), leading to identical calculation results, so that failure modes with markedly different risk characteristics are assigned the same priority ranking, making it impossible to effectively distinguish their true risk levels.
- iv. Neglect of the fuzziness of evaluation information. Traditional FMEA uses precise numerical values ranging from 1 to 10, but in actual assessments decision-makers often express themselves in natural language, which entails fuzziness and uncertainty.

Owing to the above defects, RPN ranking may mislead decision-making and reduce the credibility of FMEA. To overcome the limitations of the traditional RPN method, researchers in recent years have classified the priority ranking of failure mode risks in FMEA as a multi-attribute decision-making (MCDM) problem, employing various MCDM methods to achieve more reasonable risk ranking [1].

Among these, the TOPSIS method is one of the most widely used ranking methods, often combined with AHP to determine the weights of risk factors, and can also be used in conjunction with various subjective weighting methods and decision-maker evaluation aggregation methods to enhance the rationality of ranking results [6]. Some studies have combined fuzzy TOPSIS with fuzzy AHP and applied it to decision selection or method evaluation in various fields [8–10], while other studies have extended it to interval type-2 fuzzy [11], intuitionistic fuzzy [12] to better handle uncertain information. The VIKOR method is also a commonly used ranking method, likewise frequently employed in combination with AHP [13–15]. This method has been successfully applied in multiple fields, demonstrating good applicability. In addition, various other MCDM methods have been widely used for priority ranking of failure modes within the FMEA framework, including PROMETHEE [16], ARAS [17], MULTIMOORA [18], TODIM [19], MABAC [20], and CoCoSo [21].

However, in application, these methods generally require the prior construction of a decision matrix and the adoption of different types of uncertain numbers to characterize the evaluation values of risk attributes, primarily to overcome the limitation of traditional FMEA in expressing expert judgments with precise numerical values. The statistics of related literature are shown in Table 1.

Based on the above studies, several issues remain in the application of multi-attribute decision-making (MADM):

(1) Insufficient characterization of original information. Although existing MADM methods have introduced fuzzy set theory, models such as IFS, PyFS, and q-ROFS characterize only membership and non-membership degrees; PLTS and PULTS, despite incorporating probabilistic linguistic terms, remain confined to two-dimensional fuzziness and are unable to express neutral information. Moreover, Aleksić *et al.*, [6] pointed out that even with IT2FS or IFS, existing models still struggle to effectively distinguish between evaluations in which experts are highly certain and those in which experts are highly hesitant—that is, existing methods cannot characterize decision-makers' confidence in their own judgments. This deficiency is particularly pronounced when dealing with highly complex FMEA subjective evaluations under incomplete information, causing evaluations of differing confidence levels to be homogenized and thereby affecting the ranking of risk evaluation values.

(2) Weighting relies on subjective approaches and is limited in form. Existing studies mostly rely on subjective weighting methods such as AHP and BWM to determine the weights of risk factors, making the results susceptible to decision-makers' personal experience and leaving the weighting process itself lacking an objective basis. Existing methods cannot dynamically adjust weight allocation strategies according to decision-makers' differing tendencies to focus on the degree of attribute deviation [22]. Consequently, existing weighting approaches exhibit clear limitations in flexibly reflecting diverse decision preferences.

Table 1

Statistics on the Application of Multi-Attribute Decision-Making Methods

Author (Year)	Environment	Specific Method	Research Problem
Mangeli <i>et al.</i> [10]	Triangular fuzzy	TOPSIS, nonlinear model	Improvement of risk assessment methods
Khodadadi-Karimvand & Shirouyehzad [8]	Triangular fuzzy	TOPSIS, FMEA	Drilling risk assessment
Boral <i>et al.</i> [11]	Interval type-2 fuzzy environment	TOPSIS	Reverse supply chain risk assessment
Kushwaha <i>et al.</i> [12]	Intuitionistic fuzzy environment	TOPSIS, AHP	Power plant performance evaluation
Sudžum <i>et al.</i> [13]	Intuitionistic fuzzy environment	VIKOR, AHP	Overproduction assessment in manufacturing
Aleksić <i>et al.</i> [14]	Type-2 fuzzy environment	VIKOR, BWM	Manufacturing process failure assessment
Li <i>et al.</i> [15]	Triangular fuzzy	VIKOR, FMEA	Safety assessment of the South-to-North Water Diversion Project
Lian <i>et al.</i> [16]	Triangular fuzzy	PROMETHEE II	Failure assessment in electronic product manufacturing
Sabripoor <i>et al.</i> [17]	Triangular fuzzy	ARAS, FMEA	Surgical risk assessment
Li <i>et al.</i> [18]	Interval-valued Pythagorean fuzzy	Extended MULTIMOORA	Emergency risk assessment
He <i>et al.</i> [19]	Probabilistic linguistic information	TODIM	Wind turbine risk assessment
Aydin <i>et al.</i> [20]	Picture fuzzy	BWM, MABAC	Oil and gas safety risk assessment
Ghoushchi <i>et al.</i> [21]	Spherical fuzzy	SWARA, CoCoSo	Wind turbine risk assessment

(3) Insufficient flexibility of aggregation strategies. When aggregating evaluation information, MADM methods typically adopt fixed forms of weighted or geometric aggregation, making it difficult to adapt to the actual requirements of different decision scenarios. Gong *et al.*, [22] proposed that the form of the aggregation operator can be transformed by adjusting the parameter λ to adapt to different decision scenarios, but adjustment along a single dimension remains limited. To this end, coordinated adjustment along both the dimensions of weight allocation and aggregation form is required, so as to flexibly respond to decision-makers' differing tendencies to focus on the degree of attribute deviation.

Multi-attribute decision-making has been widely applied in various modern fields, with particular attention paid to characterizing the fuzziness and hesitancy of evaluation information. To overcome the limitation of classical fuzzy sets, which contain only membership information [23], Atanassov [24]

proposed the IFS, which simultaneously considers membership and non-membership degrees, though it is constrained by the condition that their sum not exceed 1. By relaxing this constraint, Yager [25, 26] successively proposed the PyFS and the more generalized q-ROFS, thereby enhancing the capacity to represent fuzzy information. However, the above models cannot express "neutral" information in decision-making. To this end, Cuong and Kreinovich [27] proposed the PFS, which incorporates membership, neutrality, and non-membership degrees. Mahmood *et al.*, [28] further generalized this and proposed the SFS and TSFS, where the TSFS satisfies the condition that the sum of the q -th powers of the three degrees is less than or equal to 1, offering experts greater expressive flexibility. In terms of qualitative evaluation, Zadeh [29] proposed the LTS. Rodriguez *et al.*, [30] proposed the HFLTS, which allows multiple linguistic terms to be used simultaneously but cannot reflect preferences. Pang *et al.* proposed the PLTS, which introduces probabilistic information and effectively avoids the loss of original evaluation information. Probabilistic linguistic information can effectively capture fuzziness, hesitancy, and uncertainty [31]. Further extensions gave rise to the PULIFS [32] and the PULq-ROFS [33], respectively.

In processing information, these methods rely on distance measures to characterize the deviation of alternatives from the ideal solution. Combining distance measures with ordered aggregation operators has formed a new class of information aggregation approaches that can serve as an auxiliary basis for ranking alternatives, thereby reflecting the degree of deviation between alternatives and the ideal solution more flexibly. Researchers first employed distance metrics directly as the aggregation variable of the ordered weighted averaging (OWA) operator, and further extended these to continuous data environments [34]. Incorporating probabilistic and uncertain information gave rise to the POWAD, UPOWAD, UIMOWAD, and IUEOWAD operators. The introduction of linguistic terms and intuitionistic fuzziness extended operators such as FLIEOWAD [36], FLIOWADM [37], 2LIGOWAD [38], AILOWAD [39], and IVIFOWGD [40]. With the application of higher-order fuzzy sets, operators such as TrPFLECOWMD and PLTOWD were further spawned [41, 42]. Unlike the above approaches that treat distance as an aggregated value, the distance-induced OWA (DIOWA) operator treats distance as a ranking-inducing variable, thereby preserving the original attribute values during aggregation [43]; its generalized forms, HDIOWA, NHDIOWA, and WHDIOWA, have also been systematically developed [22]. Table 2 presents the statistics of the related literature.

To meet the assessment requirements of FMEA in highly uncertain and multi-attribute fuzzy decision-making scenarios, and to further improve the complete characterization and flexible aggregation of risk information, this paper proposes the Probabilistic Uncertain Linguistic T-Spherical Fuzzy Set (PULTSFS) based on the above research deficiencies, and applies it to FMEA risk assessment. The core advantages and contributions are as follows:

(1) Complete representation of three-dimensional fuzzy information. To address the deficiency that two-dimensional models cannot express neutral information, PULTSFS simultaneously characterizes three dimensions—membership degree, neutrality degree, and non-membership degree—thereby achieving independent expression of the three attitudes of support, neutrality, and opposition, and remedying the dimensional deficiency of models such as IFS, PyFS, and q-ROFS in information characterization. On this basis, PULTSFS further introduces probabilistic linguistic terms within the three-dimensional fuzzy framework, assigning probability information to each linguistic term and thereby achieving a complete characterization of the confidence level of expert evaluations.

(2) Objective and diverse weight calculation methods. This paper proposes the Probabilistic Uncertain Linguistic T-Spherical Fuzzy Induced Ordered Weighted Distance Averaging (PULTSFDIOWA) operator under the PULTSF environment, which employs the Hamming distance from each failure mode to the positive ideal solution as the inducing variable; this distance value reflects the overall

degree of deviation between the attribute values of a failure mode and the corresponding attribute values of the ideal solution. On this basis, three weight calculation methods based on distance distribution are introduced, whereby attribute weights can be determined through three allocation strategies—decreasing weights, increasing weights, and equal weights—thereby flexibly accommodating decision-makers' preferences. Since the weights are objectively generated from the degree of deviation between attribute values and the ideal solution, there is no need to rely on subjective weighting methods, thus avoiding interference from decision-makers' personal experience in the weighting results.

Table 2
Statistics on Distance Operators

Author (Year)	Environment	Specific Method
Zhou, Chen & Liu [34]	COWD	distance measures, interval distance, continuous, OWA
Zeng, Merigó & Su [35]	UPOWAD.	interval-numbers, Hamming-distance, probability, -OWA.
Xian & Sun [36]	FLIEWAD	Atanassov's intuitionistic linguistic information, Hamming distance, OWA
Xian <i>et al.</i> [37]	FLIOWADM	fuzzy linguistic information, Minkowski distance, IOWA
Li, Zeng & Pan [38]	2LIGOWAD	2-tuple linguistic information, distance measures, generalized means, IOWA
Su <i>et al.</i> [39]	AILOWAD	Atanassov's intuitionistic linguistic information, Hamming distance, OWA
Liu & Peng [40]	IVIFOWGD	interval-valued intuitionistic fuzzy information, distance measures, OW
Xian <i>et al.</i> [41]	TrPFLECOWMD	trapezoidal Pythagorean fuzzy linguistic information, Minkowski distance, entropic measure, combined ordered weighted
Liu <i>et al.</i> [42]	PLTOWD	probabilistic linguistic term information, Hamming distance, OWA
Gong <i>et al.</i> [43]	DIOWA	distance measures as order-inducing variables, OWA
Gong <i>et al.</i> [22]	HDIOWA /	generalized DIOWA with Hamming distance / normalized Hamming distance / weighted Hamming distance
	NHDIOWA /	
	WHDIOWA	

(3) Dual-parameter adjustment of aggregation strategies. To address the limitation that existing methods adjust aggregation strategies through only a single parameter, the PULTSFDIOWA operator can not only transform the aggregation form by adjusting the parameter λ , but also simultaneously combine three different weight allocation methods—decreasing weights, increasing weights, and equal weights—thereby achieving coordinated adjustment along the two dimensions of aggregation form and weight allocation, flexibly responding to decision-makers' differing tendencies to focus on the degree of attribute deviation, and satisfying the requirements of more diverse decision scenarios.

The remainder of this paper is organized as follows: Section 2 introduces the fundamental theories and related concepts, including probabilistic uncertain linguistic sets, induced ordered weighted distance operators, and related distance measures. Section 3 introduces the content of PULTSF and proposes the probabilistic uncertain linguistic T-spherical fuzzy induced generalized ordered weighted distance operator, analyzes its basic properties, the determination method of associated weights, and its extended forms, and discusses special cases and families of the operator. Section 4 constructs a multi-attribute decision-making framework for FMEA risk assessment based on this operator, and provides a complete assessment process and implementation steps. Section 5 applies the proposed model to a case study of failure mode risk assessment for shared electric vehicles, and verifies the effectiveness and robustness of the method through comparative analysis

and sensitivity analysis. Finally, the full text is summarized and future research directions are discussed.

2. Preliminaries

This paper briefly introduces the relevant concepts of the study, including PULTS and TSFS, the OWD operator, the IOWA operator and its extended forms, and the Hamming distance.

2.1 PULTS, TSFS

Panget *al.* first defined the concept of the Probabilistic Linguistic Term Set (PLTS), which is obtained by combining hesitant fuzzy linguistic term sets with probabilistic information so as to preserve the given linguistic information to the greatest extent.

Definition 1 [57]. Let $S_{[0,k-1]}$ be an LTS. A PLTS is defined as

$$L(p) = \{s^{(t)}(p^{(t)}) | s^{(t)} \in S_{[0,k-1]}, p^{(t)} \geq 0, t=1,2,\dots,\#T, \sum_{t=1}^{\#T} p^{(t)} \leq 1\} \quad (1)$$

Where $s^{(t)}(p^{(t)})$ denotes the linguistic term $s^{(t)}$ associated with the probability $p^{(t)}$, $\#T$ is the number of all distinct linguistic terms in $L(p)$.

To express decision-makers' concerns more precisely, Lin et al., [34] proposed the concept of the Probabilistic Uncertain Linguistic Term Set (PULTS) on the basis of Uncertain Linguistic Variables (ULVs) and Probabilistic Linguistic Term Sets (PLTS).

Definition 1. A PULTS is defined as follows:

$$UL(p) = \{ \langle [s_{\alpha}^{(t)}, s_{\beta}^{(t)}], p^{(t)} \rangle | p^{(t)} \geq 0, t=1,2,\dots,\#T, \sum_{t=1}^{\#T} p^{(t)} \leq 1 \} \quad (2)$$

Where $\langle [s_{\alpha}^{(t)}, s_{\beta}^{(t)}], p^{(t)} \rangle$ is the t -th PUL element in $UL(p)$, $s_{\alpha}^{(t)}, s_{\beta}^{(t)}$ are the lower and upper bound linguistic terms, respectively, and $p^{(t)}$ is the corresponding probability.

Compared with traditional fuzzy sets and their variants, TSFS can provide experts with a broader expression space and greater flexibility in three dimensions. Mahmood *et al.*, [29] gave the following definition:

Definition 2. [29]. Suppose $X = \{x_1, x_2, \dots, x_n\}$ is a universe of discourse. Then a TSFS A on X is an object of the following form:

$$A = \{ \langle x_j, (\mu_A(x_j), \eta_A(x_j), \nu_A(x_j)) \rangle | x_j \in X \} \quad (3)$$

Where $\mu_A(x_j), \eta_A(x_j), \nu_A(x_j): X \rightarrow [0,1]$ denote the membership degree, neutrality degree, and non-membership degree of x_j to A , respectively, and they satisfy the condition $0 \leq (\mu_A(x_j))^q + (\eta_A(x_j))^q + (\nu_A(x_j))^q \leq 1 (q \geq 1)$. For computational convenience, a triple $a_j = (\mu_j, \eta_j, \nu_j)$ is called a TSFN. The parameter q can be appropriately set; the larger the value of q , the smaller the strength of commitment, whereas the smaller the value of q , the less the hesitancy and the less the uncertainty.

2.2 The Concept Of PULTSFS

Based on the advantages of PULTS and TSFS, this paper defines a novel probabilistic fuzzy set, namely PULTSFS [44]. It not only allows experts to use multiple linguistic terms from three dimensions to express evaluation information, but also incorporates the possibility of each linguistic term. The definition of PULTSFS is given as follows:

Definition 3 [45]. Let X be a universe of discourse and $S_{[0,k-1]}$ be an LTS. Then a PULTSFS $\tilde{A}(p)$ on X is defined as

$$\tilde{A}(p) = \{ \langle x_{\varsigma}, \varphi_{\varsigma}(\tilde{p})(x_{\varsigma}), \phi_{\varsigma}(\tilde{p})(x_{\varsigma}), \psi_{\varsigma}(\tilde{p})(x_{\varsigma}) \rangle | x_{\varsigma} \in X \} \quad (4)$$

where $\varphi_{\varsigma}(\hat{p})(x_{\varsigma}) = \left\{ \left[s_{\mu_{\varsigma(t)}^L}, s_{\mu_{\varsigma(t)}^U} \right] (\hat{p}(t)) \mid s_{\mu_{\varsigma(t)}^L}, s_{\mu_{\varsigma(t)}^U} \in S_{[0,k-1]}, \hat{p}(t) \geq 0, \sum_{t=1}^{\#T} \hat{p}(t) \leq 1 \right\}$ denotes the membership degree of $x_{\varsigma} \in X$; $\phi_{\varsigma}(\tilde{p})(x_{\varsigma}) = \left\{ \left[s_{\eta_{\varsigma(r)}^L}, s_{\eta_{\varsigma(r)}^U} \right] (\tilde{p}(r)) \mid s_{\eta_{\varsigma(r)}^L}, s_{\eta_{\varsigma(r)}^U} \in S_{[0,k-1]}, \tilde{p}(r) \geq 0, \sum_{r=1}^{\#R} \tilde{p}(r) \leq 1 \right\}$ denotes the neutrality degree of $x_{\varsigma} \in X$; $\psi_{\varsigma}(\bar{p})(x_{\varsigma}) = \left\{ \left[s_{v_{\varsigma(w)}^L}, s_{v_{\varsigma(w)}^U} \right] (\bar{p}(w)) \mid s_{v_{\varsigma(w)}^L}, s_{v_{\varsigma(w)}^U} \in S_{[0,k-1]}, \bar{p}(w) \geq 0, \sum_{w=1}^{\#W} \bar{p}(w) \leq 1 \right\}$ $x_{\varsigma} \in X$, with the associated probabilities being $\hat{p}(t)$, $\tilde{p}(r)$ and $\bar{p}(w)$ respectively; for $x_{\varsigma} \in X$, they satisfy the condition $0 \leq (\max_{t=1}^{\#T} \mu_{\varsigma(t)}^U)^q + (\max_{r=1}^{\#R} \eta_{\varsigma(r)}^U)^q + (\max_{w=1}^{\#W} v_{\varsigma(w)}^U)^q \leq k^q$ ($q \geq 1$).

If $X = \{x\}$, then the PULTSFS $\tilde{A}(p)$ degenerates into a PULTSFN $\tilde{a}(p)$, that is,

$$\tilde{a}(p) = \left\{ \left[s_{\mu(t)}^L, s_{\mu(t)}^U \right] \mid \hat{p}(t) \right\}, \left\{ \left[s_{\eta(r)}^L, s_{\eta(r)}^U \right] \mid \tilde{p}(r) \right\}, \left\{ \left[s_{v(w)}^L, s_{v(w)}^U \right] \mid \bar{p}(w) \right\} \quad (5)$$

Where $s_{\mu(t)}^L, s_{\mu(t)}^U, s_{\eta(r)}^L, s_{\eta(r)}^U, s_{v(w)}^L, s_{v(w)}^U \in S_{[0,k-1]}, \sum_{t=1}^{\#T} \hat{p}(t) \leq 1, \sum_{r=1}^{\#R} \tilde{p}(r) \leq 1, \sum_{w=1}^{\#W} \bar{p}(w) \leq 1$.

Definition 4[45]. Let any two PULTSFNs be

$$\begin{aligned} \tilde{a}_1(p) &= \left\{ \left[s_{\mu_{1(t)}^L}, s_{\mu_{1(t)}^U} \right] \mid \hat{p}_{1(t)} \right\}, \left\{ \left[s_{\eta_{1(r)}^L}, s_{\eta_{1(r)}^U} \right] \mid \tilde{p}_{1(r)} \right\}, \left\{ \left[s_{v_{1(w)}^L}, s_{v_{1(w)}^U} \right] \mid \bar{p}_{1(w)} \right\}, \\ \tilde{a}_2(p) &= \left\{ \left[s_{\mu_{2(t)}^L}, s_{\mu_{2(t)}^U} \right] \mid \hat{p}_{2(t)} \right\}, \left\{ \left[s_{\eta_{2(r)}^L}, s_{\eta_{2(r)}^U} \right] \mid \tilde{p}_{2(r)} \right\}, \left\{ \left[s_{v_{2(w)}^L}, s_{v_{2(w)}^U} \right] \mid \bar{p}_{2(w)} \right\}. \end{aligned}$$

For computational convenience, they need to be normalized as follows:

(1)(Probability normalization) If $0 < \sum_{t=1}^{\#T} \hat{p}_{j(t)} < 1$ (taking the membership probability as an example, $j=1,2$), $\tilde{a}_j(p)$ is normalized to $\tilde{a}_j(p^n)$, where the probability $\hat{p}_{j(t)}^n = \hat{p}_{j(t)} / \sum_{t=1}^{\#T} \hat{p}_{j(t)}$, and the corresponding standardized PULTSFN can be described as

$$\tilde{a}_j(p^n) = \left\{ \left[s_{\mu_{j(t)}^L}, s_{\mu_{j(t)}^U} \right] \mid \hat{p}_{j(t)}^n \right\}, \left\{ \left[s_{\eta_{j(r)}^L}, s_{\eta_{j(r)}^U} \right] \mid \tilde{p}_{j(r)}^n \right\}, \left\{ \left[s_{v_{j(w)}^L}, s_{v_{j(w)}^U} \right] \mid \bar{p}_{j(w)}^n \right\} \quad (6)$$

(2)(Structural standardization) If $\#T_1 \neq \#T_2$ (taking the membership degree as an example), then some linguistic terms need to be added to the element with the smaller number, with their probabilities being 0, thereby obtaining the normalized PULTSFNs $\tilde{a}_j(p^N)$ ($j=1,2$).

Definition 5[45]. Suppose $S_{[0,k-1]}$ is an LTS, and $\tilde{a}(p) = \left\{ \left[s_{\mu(t)}^L, s_{\mu(t)}^U \right] \mid \hat{p}(t) \right\}, \left\{ \left[s_{\eta(r)}^L, s_{\eta(r)}^U \right] \mid \tilde{p}(r) \right\}, \left\{ \left[s_{v(w)}^L, s_{v(w)}^U \right] \mid \bar{p}(w) \right\}$ is a PULTSFN, where $s_{\mu(t)}^L, s_{\mu(t)}^U, s_{\eta(r)}^L, s_{\eta(r)}^U, s_{v(w)}^L, s_{v(w)}^U \in S_{[0,k-1]}$, ($t=1,2,\dots,\#T$; $r=1,2,\dots,\#R$; $w=1,2,\dots,\#W$), . The score function of $\tilde{a}(p)$ is defined as

$$Sc(\tilde{a}(p)) = s \left(\frac{1}{2} \left(1 + \left(\frac{\sum_{t=1}^{\#T} (\frac{1}{2}(\mu_{(t)}^L + \mu_{(t)}^U) \hat{p}(t))}{k \sum_{t=1}^{\#T} \hat{p}(t)} \right)^q - \left(\frac{\sum_{r=1}^{\#R} (\frac{1}{2}(\eta_{(r)}^L + \eta_{(r)}^U) \tilde{p}(r))}{k \sum_{r=1}^{\#R} \tilde{p}(r)} \right)^q - \left(\frac{\sum_{w=1}^{\#W} (\frac{1}{2}(v_{(w)}^L + v_{(w)}^U) \bar{p}(w))}{k \sum_{w=1}^{\#W} \bar{p}(w)} \right)^q \right) \right) \quad (7)$$

$\tilde{a}(p)$ is defined as

$$Ac(\tilde{a}(p)) = s \left(\left(\frac{\sum_{t=1}^{\#T} (\frac{1}{2}(\mu_{(t)}^L + \mu_{(t)}^U) \hat{p}(t))}{k \sum_{t=1}^{\#T} \hat{p}(t)} \right)^q + \left(\frac{\sum_{r=1}^{\#R} (\frac{1}{2}(\eta_{(r)}^L + \eta_{(r)}^U) \tilde{p}(r))}{k \sum_{r=1}^{\#R} \tilde{p}(r)} \right)^q + \left(\frac{\sum_{w=1}^{\#W} (\frac{1}{2}(v_{(w)}^L + v_{(w)}^U) \bar{p}(w))}{k \sum_{w=1}^{\#W} \bar{p}(w)} \right)^q \right) \quad (8)$$

Definition 6[45]. Suppose there are two arbitrary PULTSFNs $\tilde{a}_1(p)$ and $\tilde{a}_2(p)$. Then the comparison rules between them are as follows:

- If $Sc(\tilde{a}_1(p)) > Sc(\tilde{a}_2(p))$, then $\tilde{a}_1(p) > \tilde{a}_2(p)$;
- If $Sc(\tilde{a}_1(p)) < Sc(\tilde{a}_2(p))$, then $\tilde{a}_1(p) < \tilde{a}_2(p)$;
- If $Sc(\tilde{a}_1(p)) = Sc(\tilde{a}_2(p))$, then
- If $Ac(\tilde{a}_1(p)) > Ac(\tilde{a}_2(p))$, then $\tilde{a}_1(p) > \tilde{a}_2(p)$;
- If $Ac(\tilde{a}_1(p)) < Ac(\tilde{a}_2(p))$, then $\tilde{a}_1(p) < \tilde{a}_2(p)$;
- If $Ac(\tilde{a}_1(p)) = Ac(\tilde{a}_2(p))$, then $\tilde{a}_1(p) \approx \tilde{a}_2(p)$;

Definition 7[45]. Suppose $S_{[0,k-1]}$ is an LTS, and $\tilde{a}_1(p) = \langle \{[s_{\mu_1(t)}^L, s_{\mu_1(t)}^U] | \hat{p}_1(t)\}, \{[s_{\eta_1(r)}^L, s_{\eta_1(r)}^U] | \tilde{p}_1(r)\}, \{[s_{v_1(w)}^L, s_{v_1(w)}^U] | \bar{p}_1(w)\} \rangle$ and $\tilde{a}_2(p) = \langle \{[s_{\mu_2(t)}^L, s_{\mu_2(t)}^U] | \hat{p}_2(t)\}, \{[s_{\eta_2(r)}^L, s_{\eta_2(r)}^U] | \tilde{p}_2(r)\}, \{[s_{v_2(w)}^L, s_{v_2(w)}^U] | \bar{p}_2(w)\} \rangle$ are any two PULTSFNs, where $\#T_1 = \#T_2 = \#T$, $\#R_1 = \#R_2 = \#R$, $\#W_1 = \#W_2 = \#W$, and for the uncertain linguistic variable part, the function $I(\cdot)$ satisfies $I(s_\alpha) = \alpha$. Then the Hamming distance $D_H(\tilde{a}_1(p), \tilde{a}_2(p))$ between $\tilde{a}_1(p)$ and $\tilde{a}_2(p)$ is defined as:

$$D(\tilde{a}_1(p), \tilde{a}_2(p)) = \frac{1}{6} \left(\frac{\sum_{t=1}^{\#T} \left(\left| \left(I(s_{\mu_1(t)}^L) \hat{p}_1(t) \right)^q - \left(I(s_{\mu_2(t)}^L) \hat{p}_2(t) \right)^q \right| \right)}{\sum_{t=1}^{\#T} \left(\left| \left(I(s_{\mu_1(t)}^L) \hat{p}_1(t) \right)^q - \left(I(s_{\mu_2(t)}^L) \hat{p}_2(t) \right)^q \right| \right)} + \frac{\sum_{r=1}^{\#R} \left(\left| \left(I(s_{\eta_1(r)}^L) \tilde{p}_1(r) \right)^q - \left(I(s_{\eta_2(r)}^L) \tilde{p}_2(r) \right)^q \right| \right)}{\sum_{r=1}^{\#R} \left(\left| \left(I(s_{\eta_1(r)}^L) \tilde{p}_1(r) \right)^q - \left(I(s_{\eta_2(r)}^L) \tilde{p}_2(r) \right)^q \right| \right)} + \frac{\sum_{w=1}^{\#W} \left(\left| \left(I(s_{v_1(w)}^L) \bar{p}_1(w) \right)^q - \left(I(s_{v_2(w)}^L) \bar{p}_2(w) \right)^q \right| \right)}{\sum_{w=1}^{\#W} \left(\left| \left(I(s_{v_1(w)}^L) \bar{p}_1(w) \right)^q - \left(I(s_{v_2(w)}^L) \bar{p}_2(w) \right)^q \right| \right)} \right) \quad (9)$$

Theorem 1. For any PULTSFNs $\tilde{a}(p) = \langle \{[s_{\mu(t)}^L, s_{\mu(t)}^U] | \hat{p}(t)\}, \{[s_{\eta(r)}^L, s_{\eta(r)}^U] | \tilde{p}(r)\}, \{[s_{v(w)}^L, s_{v(w)}^U] | \bar{p}(w)\} \rangle$, $\tilde{a}_1(p) = \langle \{[s_{\mu_1(t)}^L, s_{\mu_1(t)}^U] | \hat{p}_1(t)\}, \{[s_{\eta_1(r)}^L, s_{\eta_1(r)}^U] | \tilde{p}_1(r)\}, \{[s_{v_1(w)}^L, s_{v_1(w)}^U] | \bar{p}_1(w)\} \rangle$ and $\tilde{a}_2(p) = \langle \{[s_{\mu_2(t)}^L, s_{\mu_2(t)}^U] | \hat{p}_2(t)\}, \{[s_{\eta_2(r)}^L, s_{\eta_2(r)}^U] | \tilde{p}_2(r)\}, \{[s_{v_2(w)}^L, s_{v_2(w)}^U] | \bar{p}_2(w)\} \rangle$, then

Non-negativity: $D(\tilde{a}_1(p), \tilde{a}_2(p)) \geq 0$;

Commutativity: $D(\tilde{a}_1(p), \tilde{a}_2(p)) = D(\tilde{a}_2(p), \tilde{a}_1(p))$;

Reflexivity: $D(\tilde{a}(p), \tilde{a}(p)) = 0$;

Proof. The proofs of properties (1)–(3) are simple and straightforward, and are therefore omitted here.

2.3 The Concept of IOWDA Operator

2.3.1 The OWD operator

Based on the related concepts of the Ordered Weighted Averaging (OWA) operator, Xu and Chen [47] proposed the Ordered Weighted Distance (OWD) operator, which is defined as follows:

Definition 8[46]. Let $\mathbf{a} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\mathbf{b} = \{\beta_1, \beta_2, \dots, \beta_n\}$ be two sets of real numbers. For any elements α_j and β_j , define the distance between them as $d(\alpha_j, \beta_j) = |\alpha_j - \beta_j|$. Then

$$OWD_{(\mathbf{a}, \mathbf{b})} = \left(\sum_{j=1}^n w_j \left(d(\alpha_{\sigma(j)}, \beta_{\sigma(j)}) \right)^\lambda \right)^{1/\lambda} \quad (10)$$

The above expression is called the Ordered Weighted Distance (OWD) operator measure between the set $\mathbf{a}$ and the set $\mathbf{b}$. Let $\lambda > 0$ and $(\sigma(1), \sigma(2), \dots, \sigma(n))$ be a permutation of $(1, 2, \dots, n)$ satisfying $d(\alpha_{\sigma(j-1)}, \beta_{\sigma(j-1)}) \geq d(\alpha_{\sigma(j)}, \beta_{\sigma(j)})$. $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector of the OWD operator, satisfying $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$.

In particular, when $\lambda = 1$, $\lambda = 2$ and $\lambda \rightarrow 0$, the OWD operator is referred to as the Ordered Weighted Arithmetic Distance (OWAD) operator, the Ordered Weighted Euclidean Distance (OWED) operator, and the Ordered Weighted Geometric Distance (OWGD) operator, respectively.

2.3.2 The IOWA operator

The IOWA operator was first proposed by Yager and Filev [59] and can be regarded as a generalized form of the OWA operator.

Definition 9[46]. Let the IOWA operator be a mapping defined on $R^n \times R^n \rightarrow R$, with an associated weight vector $W = (\omega_1, \omega_2, \dots, \omega_n)$ satisfying $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. Its aggregation formula is:

$$IOWA(\langle a_1, x_1 \rangle, \langle a_2, x_2 \rangle, \dots, \langle a_n, x_n \rangle) = \sum_{i=1}^n \omega_i b_j \quad (11)$$

Where b_j is the parameter variable a_i corresponding to the j -th largest inducing variable x_i ; and $\langle a_i, x_i \rangle$ is called an IOWA pair, with a_i being the inducing variable and x_i the parameter variable. When ties occur among the inducing variables, the parameter variables corresponding to the same inducing variable may be averaged. In addition, the IOWA operator provides a parameterized family of aggregation operators ranging from the maximum to the minimum, including the OWA operator, arithmetic mean, maximum, and minimum as special cases.

2.3.3 The IOWDA operator

The IOWDA operator is an extended form of the IOWA operator, which combines the Hamming distance with the IOWA operator. The generalized form of the IOWDA operator is constructed as follows:

Definition 10[22]. Based on any type of Hamming distance measure, the IOWDA operator of dimension n is defined as a mapping $IOWDA: R^n \times R^n \rightarrow R$, with its associated weight vector W satisfying $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. Its calculation formula is:

$$IOWDA(\langle a_1, x_1 \rangle, \langle a_2, x_2 \rangle, \dots, \langle a_n, x_n \rangle) = \sum_{j=1}^n \omega_j c_j \quad (12)$$

Where $X = (x_1, x_2, \dots, x_n)$ is the set of arguments to be aggregated, and $Y = (y_1, y_2, \dots, y_n)$ is the ideal argument set. a_i represents any type of Hamming distance between x_i and y_i , c_j is the value of x_i corresponding to the j -th largest a_j after the pairs $\langle a_1, x_1 \rangle$ are sorted in descending order of a_i .

2.4 Generalized Forms of The IOWDA Operator

2.4.1 The IGOWA operator

The IGOWA operator is a further generalization of the IOWA operator, combining the generalized mean.

Definition 11[48]. An IGOWA operator is a mapping defined on $R^n \times R^n \rightarrow R$, with an associated weight vector $W = (\omega_1, \omega_2, \dots, \omega_n)$ satisfying $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. Its aggregation formula is:

$$IGOWA(\langle a_1, x_1 \rangle, \langle a_2, x_2 \rangle, \dots, \langle a_n, x_n \rangle) = \left(\sum_{i=1}^n \omega_i (b_j)^\lambda \right)^{\frac{1}{\lambda}} \quad (13)$$

Where b_j is the parameter variable x_i corresponding to the j -th largest inducing variable a_i , and $\lambda \in (-\infty, +\infty)$. When $\lambda = 1$, we obtain the IOWA operator; when $\lambda \rightarrow 0$, we obtain the IOWG operator; when $\lambda = 2$, we obtain the IOWQA operator; and when $\lambda = -1$, we obtain the IOWHA operator.

2.4.2 The Quasi-IOWA operator

The Quasi-IOWA operator is obtained by further generalizing the IOWA operator through the quasi-arithmetic mean.

Definition 12[48]. A Quasi-IOWA operator is a mapping on $R^n \times R^n \rightarrow R$, with an associated weight vector $W = (\omega_1, \omega_2, \dots, \omega_n)$ satisfying $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. Its aggregation formula is:

$$QIOWA(\langle a_1, x_1 \rangle, \langle a_2, x_2 \rangle, \dots, \langle a_n, x_n \rangle) = g^{-1} \left(\sum_{i=1}^n \omega_i (b_j) \right) \quad (14)$$

Where b_j is the parameter variable x_i corresponding to the j -th largest inducing variable a_i , and g is a strictly continuous monotonic function. If $g(b)=b^\lambda$, the IGOWA operator is obtained; if $g(b)=b$, the IOWA operator is obtained; if $g(b)=b^2$, the IOWQA operator is obtained; and if $g(b)=b^{-1}$, the IOWHA operator is obtained.

3. The PULTSF Distance-Induced Generalized Ordered Weighted Averaging Operator

3.1 The Concept Of PULTSFDIGOWA

Inspired by the concepts of the Ordered Weighted Distance (OWD) operator and the Induced Ordered Weighted Distance (IOWDA) operator, this section constructs an Distance-Induced Generalized Ordered Weighted Averaging operator for PULTSFSs.

Let X be the set consisting of all PULTSFNs, and let $\tilde{A}(p)=\{\tilde{\alpha}_1(p),\tilde{\alpha}_2(p),\dots,\tilde{\alpha}_n(p)\}$ and $\tilde{B}(p)=\{\tilde{\beta}_1(p),\tilde{\beta}_2(p),\dots,\tilde{\beta}_n(p)\}$ be two sets of PULTSFNs. Accordingly, the PULTSF weighted distance (PULTSFWD) operator and the PULTSF Distance-Included Generalized Ordered Weighted Averaging (PULTSFDIGOWA) operator are defined as follows:

Definition 13. The n -dimensional PULTSF weighted distance operator is a mapping PULTSFWD: $R^n \times R^n \rightarrow R$, with its corresponding weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, satisfying $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Its calculation formula is:

$$\text{PULTSFWD}(\tilde{A}(p), \tilde{B}(p)) = \left(\sum_{j=1}^n \omega_j \left(D(\tilde{\alpha}_j(p), \tilde{\beta}_j(p)) \right)^\lambda \right)^{1/\lambda}, \lambda > 0 \quad (15)$$

Definition 14. The n -dimensional PULTSF weighted distance operator is a mapping PULTSFDIGOWA: $R^n \times R^n \rightarrow R$, with its corresponding weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, satisfying $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Its calculation formula is:

$$R = \text{PULTSFDIGOWA}(\tilde{A}(p), \tilde{B}(p)) = \left(\sum_{j=1}^n \omega_j a_{\sigma(j)}^\lambda \right)^{1/\lambda}, \lambda > 0 \quad (16)$$

This operator takes the distance from each alternative to the PIS as the inducing variable, determines the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, and takes the corresponding score function value $a_{\sigma(j)}$ as the original attribute value.

3.2. The Properties Of PULTSFDIGOWA

Let $\langle e_i, x_i \rangle$ be the PULTSFDIGOWA data pair corresponding to operator f , where f represents any type of PULTSFDIGOWA operator described above. The relevant properties are as follows:

Theorem 2. Let $\langle \tilde{e}_1, \tilde{x}_1 \rangle, \langle \tilde{e}_2, \tilde{x}_2 \rangle, \dots, \langle \tilde{e}_n, \tilde{x}_n \rangle$ be any permutation of the PULTSFDIGOWA data pairs $\langle e_1, x_1 \rangle, \langle e_2, x_2 \rangle, \dots, \langle e_n, x_n \rangle$. Then $f(\langle \tilde{e}_1, \tilde{x}_1 \rangle, \langle \tilde{e}_2, \tilde{x}_2 \rangle, \dots, \langle \tilde{e}_n, \tilde{x}_n \rangle) = f(\langle e_1, x_1 \rangle, \langle e_2, x_2 \rangle, \dots, \langle e_n, x_n \rangle)$. This property indicates that the aggregation result of the PULTSFDIGOWA operator is independent of the original order of the input data pairs and is determined solely by the ranking rule of the inducing variables.

Theorem 3. Let $\langle D(x_1, y_1), x_1 \rangle, \langle D(x_2, y_2), x_2 \rangle, \dots, \langle D(x_n, y_n), x_n \rangle$ be PULTSFDIGOWA data pairs. Then $f(\langle D(y_1, x_1), x_1 \rangle, \langle D(y_2, x_2), x_2 \rangle, \dots, \langle D(y_n, x_n), x_n \rangle) = f(\langle D(x_1, y_1), x_1 \rangle, \langle D(x_2, y_2), x_2 \rangle, \dots, \langle D(x_n, y_n), x_n \rangle)$. This indicates that the aggregation result of PULTSFDIGOWA is not affected by the order of distance calculation, and exchanging the positions of the two inputs in the distance function will not change the value of the inducing variable nor affect the final result of the operator.

Theorem 4. Let $\langle \tilde{e}_1, \tilde{x}_1 \rangle, \langle \tilde{e}_2, \tilde{x}_2 \rangle, \dots, \langle \tilde{e}_n, \tilde{x}_n \rangle$ be PULTSFDIGOWA data pairs. If the rearranged position of $\tilde{e}_1$ is completely consistent with that of e_i and $x_i \geq \tilde{x}_i$ for all $i=1, 2, \dots, n$, then $f(\langle e_1, x_1 \rangle, \langle e_2, x_2 \rangle, \dots, \langle e_n, x_n \rangle) \geq f(\langle \tilde{e}_1, \tilde{x}_1 \rangle, \langle \tilde{e}_2, \tilde{x}_2 \rangle, \dots, \langle \tilde{e}_n, \tilde{x}_n \rangle)$. This property guarantees that, under the premise that the ranking of the inducing variables remains unchanged, the aggregation result increases as the input attribute values increase overall.

Theorem 5. Let $\langle e_1, x_1 \rangle, \langle e_2, x_2 \rangle, \dots, \langle e_n, x_n \rangle$ be PULTSFDIGOWA data pairs. If $x_i = a$ for all $i=1,2,\dots,n$, then $f(\langle e_1, x_1 \rangle, \langle e_2, x_2 \rangle, \dots, \langle e_n, x_n \rangle) = a$. That is, if all input attribute values of the PULTSFDIGOWA operator are the same constant a , then regardless of how the weights are allocated, the weighted average result must necessarily be this number itself.

Theorem 6. Let $\langle e_1, x_1 \rangle, \langle e_2, x_2 \rangle, \dots, \langle e_n, x_n \rangle$ be PULTSFDIGOWA data pairs. Then $\min_i \{x_i\} \leq f(\langle e_1, x_1 \rangle, \langle e_2, x_2 \rangle, \dots, \langle e_n, x_n \rangle) \leq \max_i \{x_i\}$. This indicates that the aggregation result of PULTSFDIGOWA always lies between the minimum and maximum of the input attribute values.

All the above properties are easy to prove, and the proofs are therefore omitted.

When the inducing variables take the same value, the average of the parameter values corresponding to these variables may be adopted as a substitute for the parameter values with identical inducing variables [48]. The PULTSFDIGOWA operator is a special case of the PULTSFOWA operator, and the PULTSFOWA operator is likewise a special case of the PULTSFOWA operator.

3.3. Families Of The PULTSFDIGOWA Operator

By adjusting the weight vector ω and the parameter λ , various special forms of the PULTSFDIGOWA operator can be derived. The specific form selected depends on the decision-maker's actual focus in a given decision problem.

3.2.1 Analyzing the weighting vector

The weight ω_j of the PULTSFDIGOWA operator is assigned to the j -th position after ranking the inducing variables in descending order. Based on this understanding, the following special cases can be obtained by setting different weight vectors:

(1) PULTSF maximum distance (Max-PULTSFDIGOWA): When the weight vector satisfies $\omega_1=1$ and $\omega_j=0(\forall j \neq 1)$, the aggregation result is determined entirely by the position with the largest inducing variable. This means that the decision-maker focuses only on the "attribute with the most severe deviation" and ignores all other attributes.

(2) PULTSF minimum distance (Min-PULTSFDIGOWA): When the weight vector satisfies $\omega_1=1$ and $\omega_j=0(\forall j \neq 1)$, the aggregation result is determined entirely by the position with the smallest inducing variable. This means that the decision-maker focuses only on the "attribute with the slightest deviation" and ignores all other attributes.

(3) When the weight vector satisfies $\omega_k=1$ and $\omega_j=0(\forall j \neq k)$, the aggregation result is determined solely by the inducing variable at the k -th position. This means that the decision-maker focuses only on the deviation value ranked k -th.

(4) When all weights are equal, $\omega_j=1/n(\forall j)$, the operator degenerates into the PULTSF normalized distance (PULTSFND), and the distance values at all ranking positions are treated equally.

(5) When the original order of the inducing variables happens to coincide with the order after descending arrangement, the PULTSFDIGOWA operator degenerates into the PULTSFOWA operator. In this case, the ranking positions correspond one-to-one with the original criteria, and the weights actually correspond to the original criteria rather than the ranking positions.

3.2.2 Analyzing the parameter λ

The parameter λ controls the type of distance aggregation, and different values of λ correspond to different distance measures.

(1) When $\lambda=1$, the PULTSFDIGOWA operator degenerates into the PULTSF induced ordered weighted arithmetic distance (PULTSFDIOWA) operator. In this case, the aggregation result is the

weighted arithmetic mean of the distances at each ranking position, which is suitable for scenarios where the decision-maker wishes to synthesize deviations across dimensions in an "equal-proportion" manner.

$$\text{PULTSFDIOWA}(\tilde{A}(p), \tilde{B}(p)) = \sum_{j=1}^n \omega_j a_{\sigma(j)} \quad (17)$$

(2) When $\lambda=2$, the operator degenerates into the PULTSF induced ordered weighted Euclidean distance average (PULTSFDIOWEA) operator. In this case, the aggregation result is the square root of the weighted arithmetic mean of the squared distances at each ranking position. This form amplifies the influence of larger deviations and is suitable for scenarios where the decision-maker wishes to give higher sensitivity to dimensions with "moderate deviation."

$$\text{PULTSFDIOWEA}(\tilde{A}(p), \tilde{B}(p)) = \left(\sum_{j=1}^n \omega_j a_{\sigma(j)}^2 \right)^{1/2} \quad (18)$$

(3) When $\lambda=1/2$, the operator degenerates into the PULTSF induced ordered weighted square root distance average (PULTSFDIOWSRA) operator. In this case, the aggregation result is the square of the weighted arithmetic mean of the square roots of the distances at each ranking position. This form compresses the influence of larger deviations and is suitable for scenarios where the decision-maker wishes to give lower sensitivity to dimensions with "moderate deviation."

$$\text{PULTSFDIOWSRA}(\tilde{A}(p), \tilde{B}(p)) = \left(\sum_{j=1}^n \omega_j a_{\sigma(j)}^{1/2} \right)^2 \quad (19)$$

(4) When $\lambda \rightarrow 0$, the operator degenerates into the PULTSF induced ordered weighted geometric distance average (PULTSFDIOWGA) operator. In this case, the aggregation result takes the weighted geometric mean form of the distances at each ranking position. This form greatly amplifies the influence of deviations and is suitable for scenarios where the decision-maker wishes to give higher sensitivity to dimensions with "severe deviation." This form amplifies the influence of larger deviations and is suitable for scenarios where the decision-maker wishes to give higher sensitivity to dimensions with "severe deviation."

$$\text{PULTSFDIOWGA}(\tilde{A}(p), \tilde{B}(p)) = \prod_{j=1}^n (a_{\sigma(j)})^{\omega_j} \quad (20)$$

According to *Theorem 6*, to satisfy the computational environment, Eq (20) can be equivalently expressed as:

$$R = e^{\sum_{j=1}^n \omega_j \ln(a_{\sigma(j)})} \quad (21)$$

Example. The process of detail aggregation using the PULTSFDIOWA operator is as follows. Let the linguistic term set be $S_{[0,8]} = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$, $q=3$, $W = (0.8, 0.2)$, and let $A = \{\tilde{a}_1(p), \tilde{a}_2(p)\}$ and $B = \{\tilde{\beta}_1(p), \tilde{\beta}_2(p)\}$ with all evaluation criteria being benefit-type criteria.

$$\begin{aligned} \tilde{a}_1(p) &= \left(\begin{array}{l} \{[s_2, s_3] | 0.5, [s_3, s_4] | 0.5\}, \\ \{[s_1, s_2] | 0.4, [s_2, s_3] | 0.6\}, \\ \{[s_0, s_1] | 0.7, [s_1, s_2] | 0.3\} \end{array} \right) \text{ and } \tilde{a}_2(p) = \left(\begin{array}{l} \{[s_2, s_3] | 0.2, [s_3, s_4] | 0.8\}, \\ \{[s_2, s_3] | 0.3, [s_3, s_4] | 0.7\}, \\ \{[s_5, s_6] | 0.4, [s_6, s_7] | 0.6\} \end{array} \right) \\ \tilde{\beta}_1(p) &= \left(\begin{array}{l} \{[s_2, s_3] | 0.75, [s_3, s_4] | 0.25\}, \\ \{[s_2, s_3] | 0.3, [s_3, s_4] | 0.7\}, \\ \{[s_1, s_2] | 0.4, [s_2, s_3] | 0.6\} \end{array} \right) \text{ and } \tilde{\beta}_2(p) = \left(\begin{array}{l} \{[s_6, s_6] | 0.4, [s_6, s_7] | 0.6\}, \\ \{[s_2, s_3] | 0.5, [s_3, s_4] | 0.5\}, \\ \{[s_1, s_2] | 0.4, [s_2, s_3] | 0.6\} \end{array} \right) \end{aligned}$$

$$\text{Sc}(\tilde{a}_1(p)) = s \left(\begin{array}{l} 1 + \left(\left(\frac{1}{2} \times (2+3) \times 0.5 + \frac{1}{2} \times (3+4) \times 0.5 \right) / 9 \times (0.5+0.5) \right)^3 \\ - \left(\left(\frac{1}{2} \times (1+2) \times 0.4 + \frac{1}{2} \times (2+3) \times 0.6 \right) / 9 \times (0.4+0.6) \right)^3 \\ - \left(\left(\frac{1}{2} \times (0+1) \times 0.7 + \frac{1}{2} \times (1+2) \times 0.3 \right) / 9 \times (0.3+0.7) \right)^3 \end{array} \right) = s_{0.512}, \text{ and}$$

$$Sc(\tilde{\beta}_1(p)) = s \left(\frac{1}{2} \left(1 + \left(\left(\frac{1}{2} \times (2+3) \times 0.75 + \frac{1}{2} \times (3+4) \times 0.25 \right) / 9 \times (0.75+0.25) \right)^3 - \left(\left(\frac{1}{2} \times (2+3) \times 0.3 + \frac{1}{2} \times (3+4) \times 0.7 \right) / 9 \times (0.3+0.7) \right)^3 - \left(\left(\frac{1}{2} \times (1+2) \times 0.4 + \frac{1}{2} \times (2+3) \times 0.6 \right) / 9 \times (0.6+0.4) \right)^3 \right) \right) = s_{0.485}$$

Similarly, it can be obtained that $Sc(\tilde{\alpha}_2(p)) = s_{0.339}$, $Sc(\tilde{\beta}_2(p)) = s_{0.647}$

From the above calculations, we obtain: $Sc(\tilde{\alpha}_1(p)) < Sc(\tilde{\alpha}_2(p))$, it follows that $\tilde{\alpha}_1(p) > \tilde{\beta}_1(p)$, $\tilde{\beta}_2(p) > \tilde{\alpha}_2(p)$, and we have $PIS = (\tilde{\alpha}_1(p), \tilde{\beta}_2(p))$.

$$D_H(\tilde{\alpha}_1(p), \tilde{\alpha}_1(p)) = 0, \quad D_H(\tilde{\beta}_2(p), \tilde{\beta}_2(p)) = 0$$

$$D_H(\tilde{\alpha}_1(p), \tilde{\beta}_1(p)) = \frac{1}{6} \left(\frac{1}{2} \left(\left| (2 \times 0.5)^3 - (2 \times 0.75)^3 \right| + \left| (3 \times 0.5)^3 - (3 \times 0.75)^3 \right| + \left| (3 \times 0.5)^3 - (3 \times 0.25)^3 \right| + \left| (4 \times 0.5)^3 - (4 \times 0.25)^3 \right| \right) + \frac{1}{2} \left(\left| (1 \times 0.4)^3 - (2 \times 0.3)^3 \right| + \left| (2 \times 0.4)^3 - (3 \times 0.3)^3 \right| + \left| (2 \times 0.6)^3 - (3 \times 0.7)^3 \right| + \left| (3 \times 0.6)^3 - (4 \times 0.7)^3 \right| \right) + \frac{1}{2} \left(\left| (0 \times 0.7)^3 - (1 \times 0.4)^3 \right| + \left| (1 \times 0.7)^3 - (2 \times 0.4)^3 \right| + \left| (1 \times 0.3)^3 - (2 \times 0.6)^3 \right| + \left| (2 \times 0.3)^3 - (3 \times 0.6)^3 \right| \right) \right) = 4.326$$

$$D_H(\tilde{\alpha}_2(p), \tilde{\beta}_2(p)) = 23.180$$

Since $D_H(\tilde{\alpha}_1(p), \tilde{\alpha}_1(p)) < D_H(\tilde{\alpha}_2(p), \tilde{\beta}_2(p))$, the ranking under set A is $c_1 = 0.339$, $c_2 = 0.512$ and $D_H(\tilde{\alpha}_1(p), \tilde{\beta}_1(p)) > D_H(\tilde{\beta}_2(p), \tilde{\beta}_2(p))$, so the ranking under set B is $c_1 = 0.485$, $c_2 = 0.647$

Therefore, the aggregation result is $PULTSFDIOWA(A) = 0.8 \times 0.339 + 0.2 \times 0.512 = 0.373$

Similarly, $PULTSFDIOWA(B) = 0.8 \times 0.485 + 0.2 \times 0.647 = 0.518$

Since $PULTSFDIOWA(A) < PULTSFDIOWA(B)$, alternative A should receive greater attention from the decision-maker.

4. A MCDM Model for FMEA Based on the PULTSFDIGOWA Operator

This section applies the proposed Probabilistic Uncertain Linguistic T-Spherical Fuzzy Set (PULTSFS) and the PULTSF Distance-Included Generalized Ordered Weighted Averaging (PULTSFDIGOWA) operator to Failure Mode and Effects Analysis (FMEA), constructing a complete risk assessment and priority ranking model. This model can provide more scientific and objective decision support for failure mode risk identification in complex engineering systems. The overall process of the model comprises five core steps, which are elaborated as follows:

Step 1. Construction and Standardization of the PULTSF Group Decision Matrix

Let $FM = \{FM_1, FM_2, \dots, FM_m\}$ denote the set of failure modes, and $C = \{O, S, D\}$ represent the set of criteria (attribute set), corresponding to Occurrence, Severity, and Detectability, respectively. The associated weight vector is $\omega = (\omega_1, \omega_2, \omega_3)^T$, satisfying $\omega_j \in [0, 1]$ and $\sum_{j=1}^3 \omega_j = 1$. Experts utilize Probabilistic Uncertain Linguistic Pythagorean Fuzzy Numbers (PULTSFNs) to articulate the evaluation information of each alternative under each criterion. These evaluation results are formulated into a decision matrix of probabilistic uncertain linguistic Pythagorean fuzzy information: $\tilde{D} = [\tilde{d}_{ij}(p)]_{m \times 3}$, $\tilde{d}_{ij}(p) = \left\{ \left\{ [s_{\mu_{ij(t)}^L}, s_{\mu_{ij(t)}^U}] \mid \hat{p}_{ij(t)} \right\}, \left\{ [s_{\eta_{ij(r)}^L}, s_{\eta_{ij(r)}^U}] \mid \tilde{p}_{ij(r)} \right\}, \left\{ [s_{\nu_{ij(w)}^L}, s_{\nu_{ij(w)}^U}] \mid \bar{p}_{ij(w)} \right\} \right\}$ ($i=1, 2, \dots, m$; $j=1, 2, 3$; $t=1, 2$; $r=1, 2\#$; $w=1, 2$).

Step 1.1. Probability Normalization and Structural Standardization. According to *Definition 4*, probability normalization and structural standardization are performed on each $\tilde{d}_{ij}(p)$ to ensure that the sum of probabilities equals 1 and that all PULTSFNs under the same attribute possess an identical number of linguistic terms.

Step 1.2. Attribute Type Normalization. The three risk factors in FMEA are all cost-type attributes, meaning that a larger numerical value indicates higher risk. Based on Eq (22), each PULTSFN is

converted into a cost-type representation. Consequently, the standardized probabilistic uncertain linguistic Pythagorean fuzzy decision matrix $\tilde{G} = [\tilde{g}_{ij}(p)]_{m \times 3}$ is obtained:

$$(\tilde{g}_{ij}(p))^c = \left(\left\{ \left[\begin{matrix} S_{\nu_{ij}(w)}^L \\ S_{\nu_{ij}(w)}^U \end{matrix} \right] \middle| \tilde{p}_{ij}(w) \right\}, \left\{ \left[\begin{matrix} S_{\eta_{ij}(r)}^L \\ S_{\eta_{ij}(r)}^U \end{matrix} \right] \middle| \tilde{p}_{ij}(r) \right\}, \left\{ \left[\begin{matrix} S_{\mu_{ij}(t)}^L \\ S_{\mu_{ij}(t)}^U \end{matrix} \right] \middle| \tilde{p}_{ij}(t) \right\} \right) \quad (22)$$

Step 2. Determination of the PULTSF Positive Ideal Solution (PIS)

The Positive Ideal Solution (PIS) is a decision matrix composed of the evaluation sets with the highest score function values among all failure modes under each risk attribute, representing the optimal performance, i.e., the lowest risk.

$$Sc(\tilde{g}_{ij}(p)) = s \left(\frac{1}{2} \left(1 + \left(\frac{\sum_{t=1}^2 \left(\frac{1}{2} (\mu_{ij}(t) + \mu_{ij}(t)) \right) \tilde{p}_{ij}(t)}{k \sum_{t=1}^2 \tilde{p}_{ij}(t)} \right)^q - \left(\frac{\sum_{r=1}^2 \left(\frac{1}{2} (\eta_{ij}(r) + \eta_{ij}(r)) \right) \tilde{p}_{ij}(r)}{k \sum_{r=1}^2 \tilde{p}_{ij}(r)} \right)^q - \left(\frac{\sum_{w=1}^2 \left(\frac{1}{2} (\nu_{ij}(w) + \nu_{ij}(w)) \right) \tilde{p}_{ij}(w)}{k \sum_{w=1}^2 \tilde{p}_{ij}(w)} \right)^q \right) \right) \quad (23)$$

The score function value can be calculated according to Eq (23). Subsequently, based on Definitions 5 and 6, the PIS $\tilde{r}_j^+(p)$ for the j-th risk attribute is defined as the value yielding the maximum score function value among all evaluations under that attribute, as shown in Eq (24). Thus, the overall PIS vector can be expressed as $\tilde{P}_+ = \{\tilde{r}_1^+(p), \tilde{r}_2^+(p), \dots, \tilde{r}_j^+(p), \dots, \tilde{r}_n^+(p)\}$.

$$\tilde{r}_j^+(p) = \left(\tilde{r}_{ij}(p) \middle| \max_i \{Sc(\tilde{r}_{ij}(p))\} \right) \quad (24)$$

Step 3. Calculation of Induced Variables and Weight Allocation

4.1 Calculation of Induced Variables

According to the PULTSFN Hamming distance Eq (9) in Definition 7, Eq (26) are used to sequentially calculate the distance between the evaluation value $\tilde{g}_{ij}(p)$ of each failure mode FMI under each risk attribute j and the $\tilde{r}_j^+(p)$. This distance serves as the induced variable in the subsequent aggregation process, reflecting the degree of deviation of the failure mode across various risk attributes.

In the PULTSF environment, distances are categorized into three formulas: PULTSFHD, PULTSFHND, and PULTSFWHD, representing the Hamming distance, normalized Hamming distance, and weighted Hamming distance, respectively.

$$(\tilde{g}_{ij}(p))^c = \left(\left\{ \left[\begin{matrix} S_{\nu_{ij}(w)}^L \\ S_{\nu_{ij}(w)}^U \end{matrix} \right] \middle| \tilde{p}_{ij}(w) \right\}, \left\{ \left[\begin{matrix} S_{\eta_{ij}(r)}^L \\ S_{\eta_{ij}(r)}^U \end{matrix} \right] \middle| \tilde{p}_{ij}(r) \right\}, \left\{ \left[\begin{matrix} S_{\mu_{ij}(t)}^L \\ S_{\mu_{ij}(t)}^U \end{matrix} \right] \middle| \tilde{p}_{ij}(t) \right\} \right) \quad (25)$$

$$D_H(\tilde{g}_{1j}(p), \tilde{g}_{2j}(p)) = \frac{1}{6} \left(\frac{\sum_{t=1}^2 \left(\left| \left(I(s_{\mu_{1j}(t)}^L) \tilde{p}_{1j}(t) \right)^q - \left(I(s_{\mu_{2j}(t)}^L) \tilde{p}_{2j}(t) \right)^q \right| \right)}{2} + \frac{\sum_{r=1}^2 \left(\left| \left(I(s_{\eta_{1j}(r)}^L) \tilde{p}_{1j}(r) \right)^q - \left(I(s_{\eta_{2j}(r)}^L) \tilde{p}_{2j}(r) \right)^q \right| \right)}{2} + \frac{\sum_{w=1}^2 \left(\left| \left(I(s_{\nu_{1j}(w)}^L) \tilde{p}_{1j}(w) \right)^q - \left(I(s_{\nu_{2j}(w)}^L) \tilde{p}_{2j}(w) \right)^q \right| \right)}{2} \right) \quad (26)$$

4.2 Weight Allocation

The PULTSFDIOIWA operator functions as a distance-type induced ordered weighted aggregation operator. Its weights directly impact the objectivity and decision-making rationality of the distance aggregation results. Weight determination must simultaneously satisfy weight normalization, non-negativity, and design rules that integrate distance characteristics, induced variables, and decision preferences.

The weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_3)^T$ of the PULTSFDIOIWA operator must satisfy the following core constraints:

(1) Non-negativity: $\omega_j \in [0, 1]$, $j=1, 2, 3$;

(2) Normalization: $\sum_{j=1}^3 \omega_j = 1$;

(3) Relationality: Weights are bound to the ranking position of the induced variables, rather than the original attribute values.

This paper proposes three weight calculation methods for distance-induced operators [49–50], which can be directly applied to the PULTSFDIOIWA operator. Through Eq (27)-(32), a weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_3)^T$ can be derived for each failure mode under the decision-maker's preferences.

(1) Decreasing Weight

A larger distance corresponds to a smaller weight, weakening the influence of attributes with excessive deviations from the ideal value. This is suitable for robust decision-making that prefers "closeness to the ideal value." It satisfies $\omega_j \leq \omega_{j-1}$ ($j \geq 2$), with weights strictly decreasing as distance increases.

$$\omega_j = \frac{D(x_j, y_j)}{\sum_{j=1}^3 D(x_j, y_j)}, j=1, 2, 3 \quad (27)$$

(2) Increasing Weight

A larger distance corresponds to a larger weight, amplifying the deviation impact of outlier attributes. This is suitable for decision scenarios requiring the highlighting of anomalous differences. It satisfies $\omega_j \leq \omega_{j-1}$ ($j \geq 2$), with weights:

$$\omega_j = \frac{e^{-D(x_j, y_j)}}{\sum_{j=1}^3 e^{-D(x_j, y_j)}}, j = 1, 2, 3 \quad (28)$$

(3) Mean Weight

A distance closer to the overall mean corresponds to a larger weight, balancing the influence of extreme distances. This is suitable for moderate decision preferences.

$$d = \max D(x_j, y_j) \quad (29)$$

$$\dot{D}(X, Y) = \frac{1}{3} \sum_{j=1}^3 D(x_j, y_j) \quad (30)$$

$$\ddot{D}(D(x_j, y_j), \dot{D}(X, Y)) = |D(x_j, y_j) - \dot{D}(X, Y)| \quad (31)$$

$$\omega_j = \frac{d - \ddot{D}(D(x_j, y_j), \dot{D}(X, Y))}{\sum_{j=1}^3 (d - \ddot{D}(D(x_j, y_j), \dot{D}(X, Y)))} \quad (32)$$

Note 2: The above weight calculations are performed independently for each failure mode. That is, the weight of each failure mode across attribute dimensions is derived independently based on its own distance vector. This calculation paradigm can adaptively adjust the importance of each attribute in the aggregation according to the unique deviation distribution characteristics of each failure mode, thereby more precisely reflecting the risk structure differences among distinct failure modes.

Step 4. Information Aggregation

According to Eq (33), the ranked scores are multiplied by the weights at corresponding positions and summed to obtain the comprehensive evaluation value R of the failure mode.

$$R_m = \text{PULTSFDIGOWA}(\langle D_{m1}, \text{Sc}(\tilde{g}_{m1}(p)) \rangle, \langle D_{m2}, \text{Sc}(\tilde{g}_{m2}(p)) \rangle, \langle D_{m3}, \text{Sc}(\tilde{g}_{m3}(p)) \rangle) = \left(\sum_{j=1}^3 \omega_j a_{\sigma(j)}^\lambda \right)^{1/\lambda} \quad (33)$$

Step 5. Ranking and Comparison

All failure modes are arranged in ascending order based on their R values. A smaller R value indicates a closer proximity to the positive ideal solution, signifying that the failure mode is more critical and demands prioritized attention, thus ranking higher. By specially setting the weight vector, this operator can also degenerate into forms such as Max-PULTSFDIGOWA or Max-PULTSFDIGOWA.

5. Illustrative Example

To verify the feasibility and effectiveness of the proposed method, this paper selects shared electric vehicle driving risk assessment as a practical case study for analysis. As a crucial component of urban green transportation, shared electric vehicles possess characteristics such as time-sharing leasing, high-frequency utilization, and multi-driver alternating operation. Their key components, including power batteries, drive systems, braking and steering systems, are prone to failures due to high usage intensity, delayed maintenance, and non-standard operations. This can lead to driving faults, loss of control, and even safety accidents such as fires and explosions, presenting significant driving risks and operational hazards.

This paper focuses on potential failure modes throughout the life cycle of shared electric vehicles. Centered around traditional FMEA risk factors—occurrence probability, severity, and detection difficulty—a risk assessment is conducted. The case literature identifies 24 typical failure modes in the driving system, power system, and auxiliary systems of shared electric vehicles through expert mutual evaluation and screening, involving 4 domain experts with authoritative weights. This subsection selects the top ten failure modes for method application.

Based on their professional knowledge and utilizing the linguistic term set $\text{LTSS}_{[0,8]} = \{s_0, s_1, \dots, s_8\} = \{\text{extremely low, very low, low, slightly low, medium, slightly high, high, very high, extremely high}\}$, the expert team employs Probabilistic Uncertain Linguistic Term Sets Fuzzy sets (PULTSFs) to express evaluation information. The initial PULTSF evaluation matrix is presented in Table 3.

Table 3
Initial PULTSF Information Provided by Experts [51]

	O	S	D		O	S	D
FM ₁	$\{[s_3, s_4] 0.8, [s_4, s_5] 0.2\}$	$\{[s_6, s_7] 0.4, [s_7, s_8] 0.6\}$	$\{[s_5, s_6] 0.5, [s_6, s_7] 0.5\}$	FM ₆	$\{[s_3, s_3] 0.3, [s_3, s_4] 0.7\}$	$\{[s_4, s_5] 0.6, [s_5, s_6] 0.4\}$	$\{[s_0, s_1] 0.7, [s_1, s_2] 0.3\}$
	$\{[s_2, s_3] 0.5, [s_3, s_4] 0.5\}$	$\{[s_1, s_2] 0.3, [s_2, s_3] 0.7\}$	$\{[s_1, s_2] 0.4, [s_2, s_3] 0.6\}$		$\{[s_0, s_1] 0.4, [s_1, s_2] 0.6\}$	$\{[s_1, s_2] 0.7, [s_2, s_3] 0.3\}$	$\{[s_0, s_1] 0.3, [s_1, s_2] 0.7\}$
	$\{[s_2, s_3] 0.2, [s_3, s_4] 0.8\}$	$\{[s_2, s_3] 0.4, [s_3, s_4] 0.6\}$	$\{[s_1, s_2] 0.7, [s_2, s_3] 0.3\}$		$\{[s_1, s_1] 0.7, [s_1, s_2] 0.3\}$	$\{[s_2, s_3] 0.3, [s_3, s_4] 0.7\}$	$\{[s_2, s_3] 0.5, [s_3, s_4] 0.5\}$
	$\{[s_1, s_1] 0.5, [s_1, s_2] 0.5\}$	$\{[s_1, s_2] 0.5, [s_2, s_3] 0.5\}$	$\{[s_1, s_1] 0.6, [s_1, s_2] 0.4\}$		$\{[s_0, s_1] 0.9, [s_1, s_2] 0.1\}$	$\{[s_2, s_3] 0.8, [s_3, s_4] 0.2\}$	$\{[s_5, s_5] 0.5, [s_5, s_6] 0.5\}$
FM ₂	$\{[s_1, s_2] 0.6, [s_2, s_3] 0.4\}$	$\{[s_0, s_1] 0.3, [s_1, s_2] 0.7\}$	$\{[s_2, s_3] 0.6, [s_3, s_4] 0.4\}$	FM ₇	$\{[s_1, s_2] 0.6, [s_2, s_3] 0.4\}$	$\{[s_0, s_0] 0.4, [s_0, s_1] 0.6\}$	$\{[s_2, s_3] 0.4, [s_3, s_4] 0.6\}$
	$\{[s_1, s_2] 0.7, [s_2, s_3] 0.3\}$	$\{[s_2, s_3] 0.7, [s_3, s_4] 0.3\}$	$\{[s_4, s_5] 0.5, [s_5, s_6] 0.5\}$		$\{[s_2, s_2] 0.6, [s_2, s_3] 0.4\}$	$\{[s_1, s_2] 0.5, [s_2, s_3] 0.5\}$	$\{[s_1, s_2] 0.8, [s_2, s_3] 0.2\}$

Table 3
Continued

	O	S	D		O	S	D
FM ₃	{{[s ₀ ,s ₁] 0.7,[s ₁ ,s ₂] 0.3},	{{[s ₂ ,s ₃] 0.3,[s ₃ ,s ₄] 0.7},	{{[s ₅ ,s ₆] 0.5,[s ₆ ,s ₇] 0.5},	FM ₈	{{[s ₃ ,s ₄] 0.1,[s ₄ ,s ₅] 0.9},	{{[s ₀ ,s ₁] 0.5,[s ₁ ,s ₂] 0.5},	{{[s ₀ ,s ₁] 0.8,[s ₁ ,s ₂] 0.2},
	{{[s ₁ ,s ₂] 0.3,[s ₂ ,s ₃] 0.7},	{{[s ₁ ,s ₂] 0.2,[s ₂ ,s ₃] 0.8},	{{[s ₂ ,s ₃] 0.8,[s ₃ ,s ₄] 0.2},		{{[s ₀ ,s ₀] 0.6,[s ₀ ,s ₁] 0.4},	{{[s ₂ ,s ₃] 0.7,[s ₃ ,s ₄] 0.3},	{{[s ₀ ,s ₀] 0.5,[s ₀ ,s ₁] 0.5},
	{{[s ₃ ,s ₄] 0.4,[s ₄ ,s ₅] 0.6}}	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6}}	{{[s ₁ ,s ₁] 0.6,[s ₁ ,s ₂] 0.4}}		{{[s ₁ ,s ₂] 0.4,[s ₂ ,s ₂] 0.6}}	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6}}	{{[s ₄ ,s ₅] 0.6,[s ₅ ,s ₅] 0.4}}
FM ₄	{{[s ₀ ,s ₀] 0.8,[s ₀ ,s ₁] 0.2},	{{[s ₅ ,s ₆] 0.3,[s ₆ ,s ₆] 0.7},	{{[s ₀ ,s ₁] 0.8,[s ₁ ,s ₂] 0.2},	FM ₉	{{[s ₁ ,s ₁] 0.5,[s ₂ ,s ₂] 0.5},	{{[s ₃ ,s ₄] 0.6,[s ₄ ,s ₅] 0.4},	{{[s ₄ ,s ₄] 0.5,[s ₄ ,s ₅] 0.5},
	{{[s ₂ ,s ₂] 0.6,[s ₂ ,s ₃] 0.4},	{{[s ₀ ,s ₁] 0.5,[s ₁ ,s ₂] 0.5},	{{[s ₁ ,s ₂] 0.4,[s ₂ ,s ₃] 0.6},		{{[s ₃ ,s ₄] 0.8,[s ₄ ,s ₅] 0.2},	{{[s ₁ ,s ₁] 0.7,[s ₁ ,s ₂] 0.3},	{{[s ₂ ,s ₃] 0.8,[s ₃ ,s ₄] 0.2},
	{{[s ₂ ,s ₃] 0.3,[s ₃ ,s ₄] 0.7}}	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6}}	{{[s ₃ ,s ₄] 0.3,[s ₄ ,s ₅] 0.7}}		{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6}}	{{[s ₂ ,s ₂] 0.2,[s ₂ ,s ₃] 0.8}}	{{[s ₁ ,s ₂] 0.7,[s ₂ ,s ₃] 0.3}}
FM ₅	{{[s ₀ ,s ₁] 0.7,[s ₁ ,s ₂] 0.3},	{{[s ₃ ,s ₄] 0.8,[s ₄ ,s ₅] 0.2},	{{[s ₃ ,s ₄] 0.2,[s ₄ ,s ₅] 0.8},	FM ₁₀	{{[s ₀ ,s ₀] 0.6,[s ₀ ,s ₁] 0.4},	{{[s ₅ ,s ₆] 0.5,[s ₆ ,s ₇] 0.5},	{{[s ₁ ,s ₂] 0.7,[s ₂ ,s ₃] 0.3},
	{{[s ₁ ,s ₂] 0.2,[s ₂ ,s ₃] 0.8},	{{[s ₂ ,s ₃] 0.2,[s ₃ ,s ₄] 0.8},	{{[s ₀ ,s ₁] 0.3,[s ₁ ,s ₂] 0.7},		{{[s ₂ ,s ₃] 0.7,[s ₃ ,s ₄] 0.3},	{{[s ₂ ,s ₃] 0.5,[s ₃ ,s ₄] 0.5},	{{[s ₀ ,s ₁] 0.6,[s ₁ ,s ₂] 0.4},
	{{[s ₁ ,s ₁] 0.5,[s ₁ ,s ₂] 0.5}}	{{[s ₁ ,s ₂] 0.3,[s ₂ ,s ₃] 0.7}}	{{[s ₁ ,s ₂] 0.5,[s ₂ ,s ₃] 0.5}}		{{[s ₄ ,s ₅] 0.6,[s ₅ ,s ₆] 0.4}}	{{[s ₁ ,s ₂] 0.3,[s ₂ ,s ₃] 0.7}}	{{[s ₃ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6}}

5.1 Application

Step 1. Based on *Definition 4*, the PULTSFNs in Table 3 are normalized and standardized. Since all three attributes are cost-type attributes, the data in Table 4 is obtained.

Table 4
Normalized and Standardized PULTSFNs

	O	S	D		O	S	D
FM ₁	{{[s ₂ ,s ₃] 0.2,[s ₃ ,s ₄] 0.8},	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6},	{{[s ₁ ,s ₂] 0.7,[s ₂ ,s ₃] 0.3},	FM ₆	{{[s ₁ ,s ₁] 0.7,[s ₁ ,s ₂] 0.3},	{{[s ₂ ,s ₃] 0.3,[s ₃ ,s ₄] 0.7},	{{[s ₂ ,s ₃] 0.5,[s ₃ ,s ₄] 0.5},
	{{[s ₂ ,s ₃] 0.5,[s ₃ ,s ₄] 0.5},	{{[s ₁ ,s ₂] 0.3,[s ₂ ,s ₃] 0.7},	{{[s ₁ ,s ₂] 0.4,[s ₂ ,s ₂] 0.6},		{{[s ₀ ,s ₁] 0.4,[s ₁ ,s ₂] 0.6},	{{[s ₁ ,s ₂] 0.7,[s ₂ ,s ₃] 0.3},	{{[s ₀ ,s ₁] 0.3,[s ₁ ,s ₂] 0.7},
	{{[s ₃ ,s ₄] 0.8,[s ₄ ,s ₅] 0.2}}	{{[s ₆ ,s ₇] 0.4,[s ₇ ,s ₈] 0.6}}	{{[s ₅ ,s ₆] 0.5,[s ₆ ,s ₆] 0.5}}		{{[s ₃ ,s ₃] 0.3,[s ₃ ,s ₄] 0.7}}	{{[s ₄ ,s ₅] 0.6,[s ₅ ,s ₆] 0.4}}	{{[s ₀ ,s ₁] 0.7,[s ₁ ,s ₂] 0.3}}
FM ₂	{{[s ₁ ,s ₂] 0.7,[s ₂ ,s ₃] 0.3},	{{[s ₂ ,s ₃] 0.7,[s ₃ ,s ₄] 0.3},	{{[s ₄ ,s ₅] 0.5,[s ₅ ,s ₆] 0.5},	FM ₇	{{[s ₂ ,s ₂] 0.6,[s ₂ ,s ₃] 0.4},	{{[s ₁ ,s ₂] 0.5,[s ₂ ,s ₃] 0.5},	{{[s ₁ ,s ₂] 0.8,[s ₂ ,s ₃] 0.2},
	{{[s ₁ ,s ₂] 0.6,[s ₂ ,s ₃] 0.4},	{{[s ₀ ,s ₁] 0.3,[s ₁ ,s ₂] 0.7},	{{[s ₂ ,s ₃] 0.6,[s ₃ ,s ₄] 0.4},		{{[s ₁ ,s ₂] 0.6,[s ₂ ,s ₃] 0.4},	{{[s ₀ ,s ₀] 0.4,[s ₀ ,s ₁] 0.6},	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6},
	{{[s ₁ ,s ₁] 0.5,[s ₁ ,s ₂] 0.5}}	{{[s ₁ ,s ₂] 0.5,[s ₂ ,s ₂] 0.5}}	{{[s ₁ ,s ₁] 0.6,[s ₁ ,s ₂] 0.4}}		{{[s ₀ ,s ₁] 0.9,[s ₁ ,s ₂] 0.1}}	{{[s ₂ ,s ₃] 0.8,[s ₃ ,s ₄] 0.2}}	{{[s ₅ ,s ₅] 0.5,[s ₅ ,s ₆] 0.5}}
FM ₃	{{[s ₃ ,s ₄] 0.4,[s ₄ ,s ₅] 0.6},	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6},	{{[s ₁ ,s ₁] 0.6,[s ₁ ,s ₂] 0.4},	FM ₈	{{[s ₁ ,s ₂] 0.4,[s ₂ ,s ₂] 0.6},	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6},	{{[s ₄ ,s ₅] 0.6,[s ₅ ,s ₅] 0.4},
	{{[s ₁ ,s ₂] 0.3,[s ₂ ,s ₃] 0.7},	{{[s ₁ ,s ₂] 0.2,[s ₂ ,s ₃] 0.8},	{{[s ₂ ,s ₃] 0.8,[s ₃ ,s ₄] 0.2},		{{[s ₀ ,s ₀] 0.6,[s ₀ ,s ₁] 0.4},	{{[s ₂ ,s ₃] 0.7,[s ₃ ,s ₄] 0.3},	{{[s ₀ ,s ₀] 0.5,[s ₀ ,s ₁] 0.5},
	{{[s ₀ ,s ₁] 0.7,[s ₁ ,s ₂] 0.3}}	{{[s ₂ ,s ₃] 0.3,[s ₃ ,s ₄] 0.7}}	{{[s ₅ ,s ₆] 0.5,[s ₆ ,s ₇] 0.5}}		{{[s ₃ ,s ₄] 0.1,[s ₄ ,s ₅] 0.9}}	{{[s ₀ ,s ₁] 0.5,[s ₁ ,s ₂] 0.5}}	{{[s ₀ ,s ₁] 0.8,[s ₁ ,s ₂] 0.2}}
FM ₄	{{[s ₂ ,s ₃] 0.3,[s ₃ ,s ₄] 0.7},	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6},	{{[s ₃ ,s ₄] 0.3,[s ₄ ,s ₅] 0.7},	FM ₉	{{[s ₂ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6},	{{[s ₂ ,s ₂] 0.2,[s ₂ ,s ₃] 0.8},	{{[s ₁ ,s ₂] 0.7,[s ₂ ,s ₃] 0.3},
	{{[s ₂ ,s ₂] 0.6,[s ₂ ,s ₃] 0.4},	{{[s ₀ ,s ₁] 0.5,[s ₁ ,s ₂] 0.5},	{{[s ₁ ,s ₂] 0.4,[s ₂ ,s ₃] 0.6},		{{[s ₃ ,s ₄] 0.8,[s ₄ ,s ₅] 0.2},	{{[s ₁ ,s ₁] 0.7,[s ₁ ,s ₂] 0.3},	{{[s ₂ ,s ₃] 0.8,[s ₃ ,s ₄] 0.2},
	{{[s ₀ ,s ₀] 0.8,[s ₀ ,s ₁] 0.2}}	{{[s ₅ ,s ₆] 0.3,[s ₆ ,s ₆] 0.7}}	{{[s ₀ ,s ₁] 0.8,[s ₁ ,s ₂] 0.2}}		{{[s ₁ ,s ₁] 0.5,[s ₂ ,s ₂] 0.5}}	{{[s ₃ ,s ₄] 0.6,[s ₄ ,s ₅] 0.4}}	{{[s ₄ ,s ₄] 0.5,[s ₄ ,s ₅] 0.5}}

Table 4
Continued

	O	S	D		O	S	D
FM ₅	{{[s ₁ ,s ₁] 0.5,[s ₁ ,s ₂] 0.5},	{{[s ₁ ,s ₂] 0.3,[s ₂ ,s ₃] 0.7},	{{[s ₁ ,s ₂] 0.5,[s ₂ ,s ₃] 0.5},	FM ₁₀	{{[s ₄ ,s ₅] 0.6,[s ₅ ,s ₆] 0.4},	{{[s ₁ ,s ₂] 0.3,[s ₂ ,s ₃] 0.7},	{{[s ₃ ,s ₃] 0.4,[s ₃ ,s ₄] 0.6},
	{{[s ₁ ,s ₂] 0.2,[s ₂ ,s ₃] 0.8},	{{[s ₂ ,s ₃] 0.2,[s ₃ ,s ₄] 0.8},	{{[s ₀ ,s ₁] 0.3,[s ₁ ,s ₂] 0.7},		{{[s ₂ ,s ₃] 0.7,[s ₃ ,s ₄] 0.3},	{{[s ₂ ,s ₃] 0.5,[s ₃ ,s ₄] 0.5},	{{[s ₀ ,s ₁] 0.6,[s ₁ ,s ₂] 0.4},
	{{[s ₀ ,s ₁] 0.7,[s ₁ ,s ₂] 0.3}}	{{[s ₃ ,s ₄] 0.8,[s ₄ ,s ₅] 0.2}}	{{[s ₃ ,s ₄] 0.2,[s ₄ ,s ₅] 0.8}}		{{[s ₀ ,s ₀] 0.6,[s ₀ ,s ₁] 0.4}}	{{[s ₅ ,s ₆] 0.5,[s ₆ ,s ₇] 0.5}}	{{[s ₁ ,s ₂] 0.7,[s ₂ ,s ₃] 0.3}}

Step 2. According to the constraint condition q in *Definition 3*, which must not be less than 2.29, q is set to 3 for the following calculation steps to compute the score values of each failure mode under each evaluation dimension, thereby obtaining the ideal solution matrix PIS.

$$PIS = \begin{pmatrix} O: ([s_4, s_5]|0.6, [s_5, s_6]|0.4), [s_2, s_3]|0.7, [s_3, s_4]|0.3, [s_0, s_0]|0.6, [s_0, s_1]|0.4) \\ S: ([s_2, s_3]|0.7, [s_3, s_4]|0.3), [s_0, s_1]|0.3, [s_1, s_2]|0.7, [s_1, s_2]|0.5, [s_2, s_2]|0.5) \\ D: ([s_4, s_5]|0.6, [s_5, s_5]|0.4), [s_0, s_0]|0.5, [s_0, s_1]|0.5, [s_0, s_1]|0.8, [s_1, s_2]|0.2) \end{pmatrix}$$

Step 3. Calculate the PULTSFDIGOWA with $\lambda=1$. According to Eq (26), the induced variables are computed; based on Eq (27), the weights of each failure mode across various dimensions are calculated under the decreasing weight rule. The results are shown in Table 5.

Table 5
Calculation Results of Induced Variables

	Induced Variable			Weight Allocation		
	O	S	D	O	S	D
FM ₁	10.825	21.202	12.728	0.242	0.474	0.284
FM ₂	5.850	0.000	4.842	0.547	0.000	0.453
FM ₃	6.319	6.035	15.521	0.227	0.217	0.557
FM ₄	4.841	15.357	7.934	0.172	0.546	0.282
FM ₅	7.433	8.054	12.602	0.265	0.287	0.449
	Induced Variable			Weight Allocation		
	O	S	D	O	S	D
FM ₁	10.825	21.202	12.728	0.242	0.474	0.284
FM ₂	5.850	0.000	4.842	0.547	0.000	0.453
FM ₃	6.319	6.035	15.521	0.227	0.217	0.557
FM ₄	4.841	15.357	7.934	0.172	0.546	0.282
FM ₅	7.433	8.054	12.602	0.265	0.287	0.449
	Induced Variable			Weight Allocation		
	O	S	D	O	S	D
FM ₆	8.898	8.795	3.700	0.416	0.411	0.173
FM ₇	5.669	2.897	12.250	0.272	0.139	0.589
FM ₈	17.569	3.602	0.000	0.830	0.170	0.000
FM ₉	6.539	5.039	9.196	0.315	0.243	0.443
FM ₁₀	0.000	11.641	4.123	0.000	0.739	0.262
	Induced Variable			Weight Allocation		
	O	S	D	O	S	D
FM ₆	8.898	8.795	3.700	0.416	0.411	0.173
FM ₇	5.669	2.897	12.250	0.272	0.139	0.589
FM ₈	17.569	3.602	0.000	0.830	0.170	0.000
FM ₉	6.539	5.039	9.196	0.315	0.243	0.443
FM ₁₀	0.000	11.641	4.123	0.000	0.739	0.262

Step 4. and *Step 5.* According to Eq (33), the comprehensive evaluation values of each failure mode are calculated and ranked. Furthermore, a comparison is made between the calculation results of the PULTSF maximum distance allocation weight and the minimum distance allocation weight. The results are shown in Table 6.

Table 6
Comprehensive Evaluation Value Calculation Results

	FM ₁	FM ₂	FM ₃	FM ₄	FM ₅	FM ₆	FM ₇	FM ₈	FM ₉	FM ₁₀	Rank
PULTSFDIOWA(R)	0.3	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.39	FM ₁ > FM ₁₀ > FM ₃ > FM ₇ > FM ₄ > FM ₈ > FM ₉ > FM ₅ > FM ₆ > FM ₂
Min-PULTSFIOWDA(R)	0.2	0.4	0.3	0.3	0.4	0.4	0.3	0.4	0.4	0.34	FM ₁ > FM ₃ > FM ₁₀ > FM ₄ > FM ₇ > FM ₆ > FM ₉ > FM ₈ > FM ₅ > FM ₂
Max-PULTSFIOWDA(R)	0.4	0.5	0.5	0.5	0.4	0.5	0.5	0.5	0.4	0.56	FM ₁ > FM ₉ > FM ₅ > FM ₇ > FM ₆ > FM ₃ > FM ₄ > FM ₁₀ > FM ₂ > FM ₈

5.2 Comparison Analysis

(1) Weight comparison

The differences in PULTSFDIOWA(R) under different weight calculation methods were computed and compared with $q=3$. Figure 1 displays the obtained results.

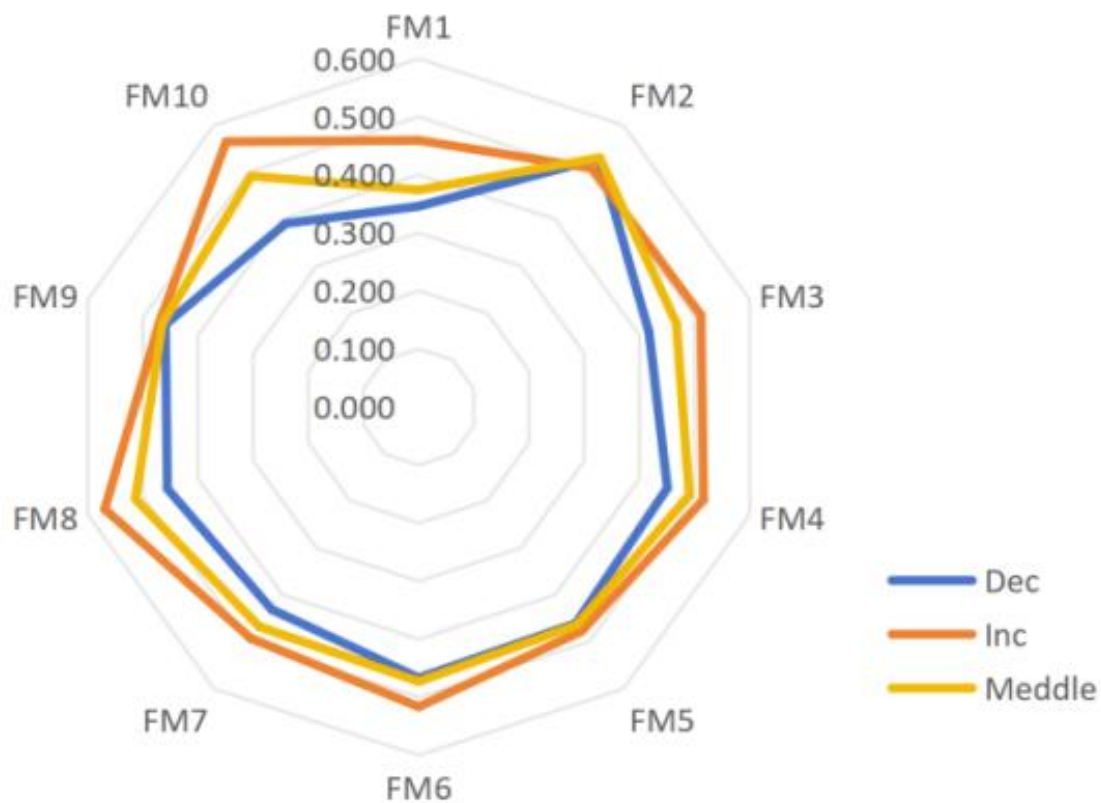


Fig. 1. Radar chart for comparison of weight allocation methods

Comparatively speaking, calculation using weight method one (the decreasing weight type) can more fully reflect the internal differences of the weight vector, enable high-weight attributes to play a greater role in the comprehensive evaluation, and ensure that the evaluation results correctly reflect the importance of each attribute according to the magnitude of the weights.

(2) Comparison of Hamming distance measures

Under the PULTSF environment and decreasing weights, the evaluation values R corresponding to the three forms of Hamming distance were calculated and their rankings compared. In the calculation under the WHD distance measure, the proportion weights of the TRW three dimensions were set as $1/2$, $1/3$, and $1/6$; when the proportion weights of these three dimensions were all $1/3$, PULTSF WHDIOWA degenerated into PULTSF NHD IOWA. The results are shown in Table 7.

Table 7
Comparison of different distance measures

	FM ₁	FM ₂	FM ₃	FM ₄	FM ₅	FM ₆	FM ₇	FM ₈	FM ₉	FM ₁₀
PULTSFDIOWA(R)	0.346	0.530	0.417	0.451	0.461	0.467	0.431	0.456	0.459	0.391
PULTSFNHD IOWA(R)	0.346	0.530	0.417	0.451	0.461	0.467	0.431	0.456	0.459	0.391
PULTSF WHDIOWA(R)	0.372	0.531	0.466	0.475	0.465	0.473	0.464	0.510	0.464	0.476
Rank										
PULTSFDIOWA(R)	FM ₁ > FM ₁₀ > FM ₃ > FM ₇ > FM ₄ > FM ₈ > FM ₉ > FM ₅ > FM ₆ > FM ₂									
PULTSFNHD IOWA(R)	FM ₁ > FM ₃ > FM ₁₀ > FM ₄ > FM ₇ > FM ₉ > FM ₈ > FM ₅ > FM ₆ > FM ₂									
PULTSF WHDIOWA(R)	FM ₁ > FM ₉ > FM ₅ > FM ₃ > FM ₆ > FM ₄ > FM ₁₀ > FM ₇ > FM ₈ > FM ₂									

In the decreasing-type and intermediate-type weight calculation methods, since the calculation formulas adopt proportional normalization and linear transformation, respectively, the weights under the HD and NHD distance measures are equal, exhibiting scaling invariance; whereas in the increasing-type method, since the weight calculation formula contains an exponential decay term with respect to distance, $\exp(-D)$, the weights vary nonlinearly with distance, and therefore the weights under the HD and NHD distance measures are not equal.

(3) Comparison of the PULTSFDIGOWA operator and its variants

The PULTSFDIGOWA operator and its variants were used to calculate the distance between the ideal solution and each failure mode, and the results under weight calculation method one and the HD distance measure are shown in the table. The results are shown in Table 8.

Table 8
PULTSFDIGOWA operator aggregated results

λ	FM ₁	FM ₂	FM ₃	FM ₄	FM ₅	FM ₆	FM ₇	FM ₈	FM ₉	FM ₁₀
$\rightarrow 0$	0.336	0.528	0.408	0.445	0.460	0.466	0.428	0.455	0.459	0.384
0.5	0.341	0.529	0.413	0.448	0.461	0.466	0.429	0.455	0.459	0.387
1	0.346	0.530	0.417	0.451	0.461	0.467	0.431	0.456	0.459	0.391
2	0.356	0.531	0.427	0.457	0.461	0.468	0.435	0.456	0.460	0.398
Rank										
$\rightarrow 0$	FM ₁ > FM ₁₀ > FM ₃ > FM ₇ > FM ₄ > FM ₈ > FM ₉ > FM ₅ > FM ₆ > FM ₂									
0.5	FM ₁ > FM ₁₀ > FM ₃ > FM ₇ > FM ₄ > FM ₈ > FM ₉ > FM ₅ > FM ₆ > FM ₂									
1	FM ₁ > FM ₁₀ > FM ₃ > FM ₇ > FM ₄ > FM ₈ > FM ₉ > FM ₅ > FM ₆ > FM ₂									
2	FM ₁ > FM ₁₀ > FM ₃ > FM ₇ > FM ₈ > FM ₄ > FM ₉ > FM ₅ > FM ₆ > FM ₂									

According to the calculation results of the PULTSFDIGOWA operator, the ranking results of each failure mode remain stable under different values of λ . This result indicates that, under this weight calculation method and the HD distance measure, the decision results are insensitive to changes in the parameter λ , and only FM₄ and FM₈ obtain different rankings when $\lambda=2$, which nevertheless exhibits good consistency.

(4) Comparison of the PULTSFQDIGOWA operator and its variants

When $q=3$, the PULTSFQDIGOWA operator and its variants were used to calculate the distance between the ideal solution and each failure mode, and the results under weight calculation method one and the HD distance measure are shown in the table. According to Definition 12, in order to satisfy the monotonic increasing property of $g(b)$, the score function values were scaled to the interval $[0, 1]$ when calculating the comprehensive evaluation values of the PULTSFQDIGOWA trigonometric function variants. The computed values are provided in Table 9.

Table 9
PULTSFQDIGOWA operator aggregated results

	FM ₁	FM ₂	FM ₃	FM ₄	FM ₅	FM ₆	FM ₇	FM ₈	FM ₉	FM ₁₀
$g\left(\sin \frac{\pi}{2} b\right)$	0.230	0.822	0.429	0.540	0.632	0.645	0.508	0.613	0.627	0.363
$g\left(\cos \frac{\pi}{2} b\right)$	0.367	0.866	0.555	0.637	0.638	0.660	0.562	0.622	0.635	0.462
$g\left(\tan \frac{\pi}{2} b\right)$	0.341	0.978	0.715	0.793	0.649	0.686	0.618	0.639	0.643	0.579
Rank										
$g\left(\sin \frac{\pi}{2} b\right)$	FM ₁ > FM ₁₀ > FM ₃ > FM ₇ > FM ₄ > FM ₈ > FM ₉ > FM ₅ > FM ₆ > FM ₂									
$g\left(\cos \frac{\pi}{2} b\right)$	FM ₁ > FM ₁₀ > FM ₃ > FM ₇ > FM ₈ > FM ₉ > FM ₄ > FM ₅ > FM ₆ > FM ₂									
$g\left(\tan \frac{\pi}{2} b\right)$	FM ₁ > FM ₁₀ > FM ₇ > FM ₈ > FM ₉ > FM ₅ > FM ₆ > FM ₃ > FM ₄ > FM ₂									

According to the calculation results of the PULTSFQDIGOWA operator, when the sine, cosine, and tangent trigonometric functions are adopted as the g function, certain differences exist in the ranking results of each failure mode. FM₁ ranks first under all functions, and FM₂ ranks last. However, the choice of trigonometric function also exerts a certain influence on the ranking results, which is closely related to the nonlinear properties of trigonometric functions on the interval $[0, 1]$. Specifically, $g\left(\sin \frac{\pi}{2} b\right)$ is a convex function on the interval, while the cosine function $g\left(\cos \frac{\pi}{2} b\right)$ is a concave function, and the two differ in their degree of emphasis on intermediate values; however, owing to the gentle variation of the curves, the amplification and compression effects on intermediate values are small, resulting in only minor differences in the ranking results. The tangent function $g\left(\tan \frac{\pi}{2} b\right)$ tends to infinity as $b \rightarrow 1^-$, exhibiting an extreme amplification effect that triggers drastic changes in the rankings of FM₃–FM₉.

5.3 Sensitivity Analysis

To examine the influence of changes in the parameter q on the ranking results of the alternatives, the parameter q was assigned values within the interval $[3, 7]$, and the comprehensive evaluation value rankings of each failure mode under the decreasing-type weights and the Hamming distance measure were analyzed. Figure 2 displays the obtained results.

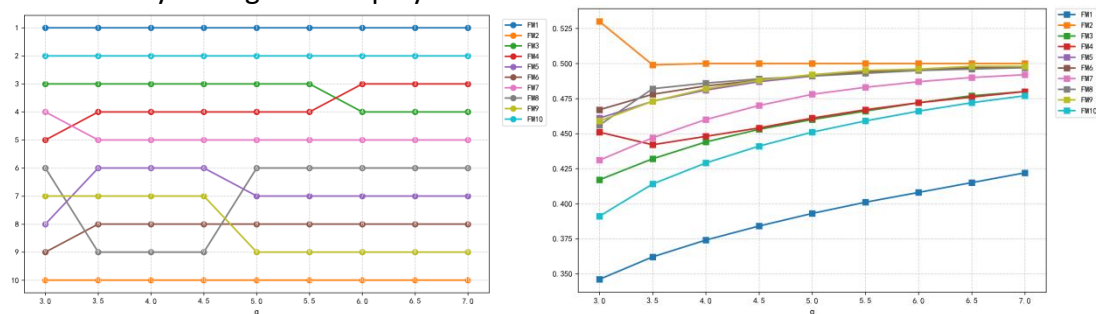


Fig. 2. Calculation results of different distance operators under different parameters q

Under the decreasing-type weights and the Hamming distance measure, when the parameter q gradually increases from 3 to 7, the comprehensive evaluation values R of all failure modes exhibit a monotonically increasing trend, but the rankings of the five failure modes always remain unchanged. From the ranking results, the overall priority conclusions of the failure modes possess strong robustness: the rankings of FM_1 (highest risk), FM_{10} (second-highest risk), and FM_2 (lowest risk) remain unchanged throughout the entire range of q , and the core risk conclusions are not affected by the parameter. Only the failure modes within the low-priority interval ($FM_3, FM_4, FM_5, FM_6, FM_7, FM_8, FM_9$) undergo local positional interchanges, and the fluctuation amplitude is limited, without altering the overall risk pattern. Under this decision scenario, the ranking results of the PULTSFDIOWA model exhibit good robustness with respect to the parameter q .

The comprehensive evaluation value rankings of each failure mode under PULTSFWDIOWA with decreasing-type weights when $q=3$ ω_T 、 ω_R and ω_W are the weights of the normalized membership, neutrality, and non-membership degrees, respectively.

(1) When $\lambda=1, q=3, \omega_T=0.2, \omega_R=0.1, \omega_W=0.7$, the influence of changes in ω_R and ω_W on the results and rankings of the PULTSFWDIOWA comprehensive evaluation value R when ω_T is fixed is discussed, as shown in Figure 3(a).

(2) When $\lambda=1, q=3, \omega_T=0.1, \omega_R=0.2, \omega_W=0.7$, the influence of changes in ω_T and ω_W on the ranking results of the PULTSFWDIOWA comprehensive evaluation value R when ω_R is fixed is discussed, as shown in Figure 3(b).

(3) When $\lambda=1, q=3, \omega_T=0.1, \omega_R=0.7, \omega_W=0.2$, the influence of changes in ω_T and ω_R on the ranking results of the PULTSFWDIOWA comprehensive evaluation value R when ω_W is fixed is discussed, as shown in Figure 3(c).

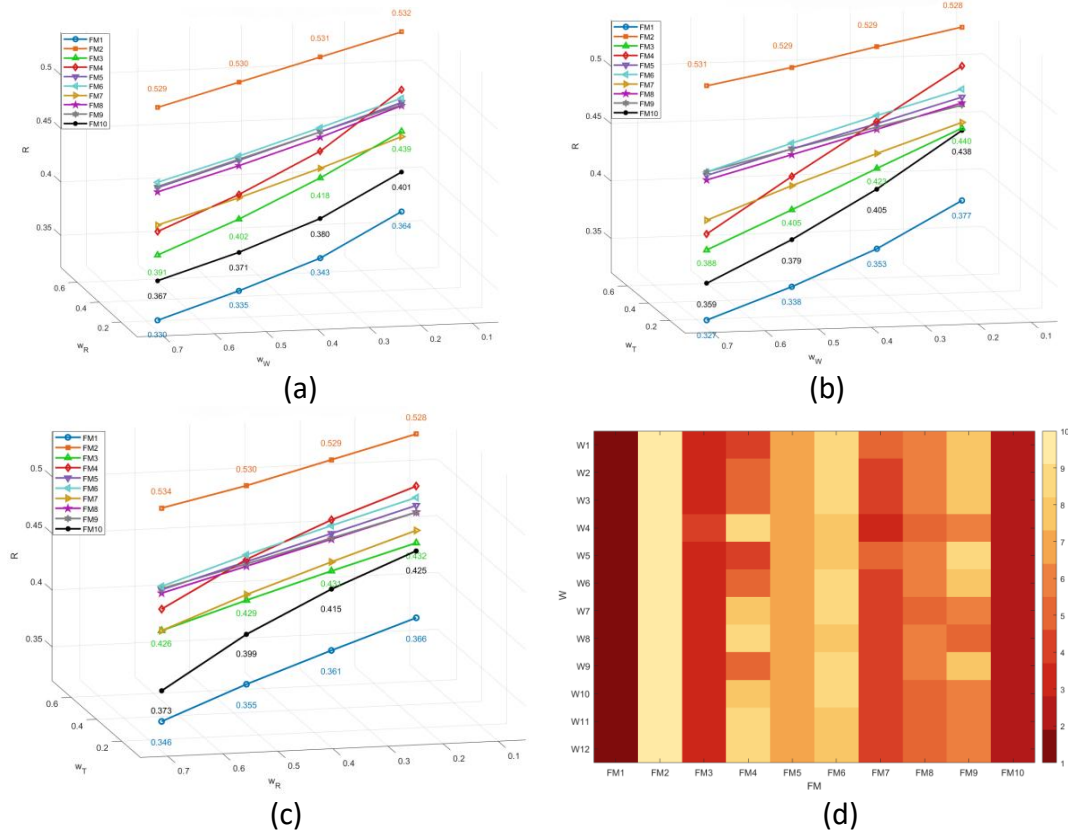


Fig. 3. Effect of weight allocation on R

In the three sets of perturbation experiments, the influence of PULTSFWHDIOWA distance weight allocation on the ranking results of failure modes was investigated. The highest-priority FM₁, the second-highest-priority FM₁₀, and the lowest-priority FM₂ consistently maintained their rankings under all weight combinations. Among FM₃–FM₉, FM₅ remained stable at rank 7 in Figure 3(a); FM₃, FM₆, FM₈, and FM₉ exhibited only slight positional adjustments; when ω_R took a relatively low value, FM₄ and FM₇ ranked within the top 5, and as ω_R increased and ω_W decreased, the overall ranking of FM₄ continued to decline, eventually falling to rank 9, while the ranking of FM₇ rose steadily, ultimately reaching rank 3. In Figure 3(b), the rankings of FM₃, FM₅, FM₆, and FM₇ remained unchanged; FM₈ and FM₉ exhibited only slight positional adjustments; as ω_T increased and ω_W gradually decreased, FM₄ dropped successively to ranks 5 and 8, and ultimately fell to rank 9, with a substantial downward shift in position. In Figure 3(c), the rankings of FM₅ and FM₈ remained stable; FM₃, FM₆, and FM₇ exhibited slight positional adjustments; as ω_T increased and ω_R gradually decreased, FM₄ finally stabilized at rank 9.

The ranking results indicate that the decision conclusions possess strong robustness against weight perturbations. The highest-priority FM₁, the second-highest-priority FM₁₀, and the lowest-priority FM₂ consistently maintained their rankings under all weight combinations. Most medium-risk failure modes exhibited only minor positional fluctuations within the same tier; only FM₄ was highly sensitive to the allocation of ω_T , ω_R , ω_W , with its risk ranking declining to rank 9 under multiple weight settings. The risk evaluation result of this failure mode is thus susceptible to the influence of decision-makers' preferences.

6. Conclusion

This paper addresses the deficiencies of traditional Failure Mode and Effects Analysis (FMEA) methods and existing fuzzy multi-attribute decision-making models by proposing the Probabilistic Uncertain Linguistic T-Spherical Fuzzy Set (PULTSFS) and constructing an FMEA risk assessment decision framework based on the PULTSFDIOWA operator, thereby achieving precise ranking and risk identification of failure modes under complex uncertain environments.

By integrating the Probabilistic Uncertain Linguistic Term Set (PULTS) with the T-Spherical Fuzzy Set (TSFS), a novel PULTSFS model is developed. This model is capable of simultaneously characterizing three-dimensional fuzzy information comprising membership, neutrality, and non-membership degrees, while incorporating probabilistic uncertain linguistic evaluations, thereby fully preserving the fuzziness, hesitancy, and uncertainty inherent in expert assessments. It effectively remedies the deficiencies of existing models, such as the absence of neutral information expression, information loss, and insufficient dimensionality, and is better aligned with practical FMEA evaluation scenarios. Subsequently, the PULTSF Distance-Induced Generalized Ordered Weighted Averaging (PULTSFDIOWA) operator is constructed, which adopts the distance from each failure mode to the positive ideal solution as the inducing variable, supports three objective weight assignment schemes—decreasing, increasing, and equal weights—and allows flexible switching of distance aggregation forms through the parameter λ . This operator can dynamically adjust weight allocation and aggregation rules according to decision-makers' preferences, thereby balancing decision robustness and flexibility. Instance verification, comparative analysis, and sensitivity analysis demonstrate that the proposed model possesses stronger information representation capability, higher risk discrimination, and better robustness, and can effectively identify critical risk factors and yield stable and reliable priority ranking of failure modes, thus providing more scientific and effective decision support for FMEA risk assessment under complex uncertain environments.

Two limitations of the method remain: (1) The model does not account for decision-makers' bounded rational behavior, including loss aversion and regret aversion tendencies, so risk evaluations may fail to reflect actual psychological preferences. (2) The model employs only the distance from each failure mode to the positive ideal solution as the ranking basis, without considering its distance to the negative ideal solution or the relative closeness, and unipolar comparison may lose negative evaluation information.

To address these limitations, future research will consider introducing bounded rationality decision theories such as loss aversion or regret aversion to more accurately reflect experts' realistic psychological preferences; meanwhile, the distance to the negative ideal solution will be incorporated into the inducing variables, and a comprehensive closeness index based on both positive and negative ideal solutions will be constructed, or a dual-induced ranking approach based on positive and negative distances will be adopted to aggregate bidirectional deviation information and enhance the discrimination capability of risk evaluation.

Acknowledgement

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Conflicts of Interest

The authors declare no conflicts of interest.

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